

# Polarized QED splittings of massive fermions and dipole subtraction for non-collinear-safe observables

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## Abstract:

Building on earlier work, the dipole subtraction formalism for photonic corrections is extended to various photon–fermion splittings where the resulting collinear singularities lead to corrections that are enhanced by logarithms of small fermion masses. The difference to the earlier treatment of photon radiation is that now no cancellation of final-state singularities is assumed, i.e. we allow for non-collinear-safe final-state radiation. Moreover, we consider collinear fermion production from incoming photons, forward-scattering of incoming fermions, and collinearly produced fermion–antifermion pairs. For all cases we also provide the corresponding formulas for the phase-space slicing approach, and particle polarization is supported for all relevant situations. A comparison of numerical results obtained with the proposed subtraction procedure and the slicing method is explicitly performed for the sample process  $e^- \gamma \rightarrow e^- \mu^- \mu^+$ .

# 1 Introduction

Present and future collider experiments require precise predictions for particle reactions, i.e. for most of the relevant processes radiative corrections have to be calculated. This task becomes arbitrarily complicated if either the order in perturbation theory (loop level) or the number of external particles is increased, or both. In recent years the needed techniques and concepts have received an enormous boost from various directions; for a brief overview we refer to some recent review articles [1,2].

In this paper we focus on real emission corrections involving photons at next-to-leading order (NLO). Apart from the integration over a many-particle phase space, here the main complication is the proper isolation of the singular parts which originate from soft or collinear regions in phase space. To solve this problem at NLO, two different types of methods have been developed in the past: *phase-space slicing* (see, e.g., Ref. [3]) and *subtraction* [4–8] techniques. In the slicing approach the singular regions are cut off from phase space in the numerical integration and treated separately. Employing general factorization properties of squared amplitudes in the soft or collinear regions, the singular integrations can be carried out analytically. In the limit of small cutoff parameters the sum of the two contributions reproduces the full phase-space integral. There is a trade-off between residual cut dependences and numerical integration errors which increase with decreasing slicing cuts; in practice, one is forced to search for a plateau in the integrated result within some errors by varying the slicing cut parameters.

This cumbersome procedure is not necessary within subtraction formalisms which are based on the idea of subtracting a simple auxiliary function from the singular integrand and adding this contribution again. This auxiliary function has to be chosen in such a way that it cancels all singularities of the original integrand so that the phase-space integration of the difference can be performed numerically, even over the singular regions of the original integrand. In this difference the original matrix element can be evaluated without regulators for soft or collinear singularities. The auxiliary function has to be simple enough so that it can be integrated over the singular regions analytically with the help of regulators, when the subtracted contribution is added again. This singular analytical integration can be done once and for all in a process-independent way because of the general factorization properties of squared amplitudes in the singular regions. At NLO several subtraction variants have been proposed in the literature [4–8], some of which are quite general; at next-to-next-to-leading order subtraction formalisms are still under construction [9].

The *dipole subtraction formalism* certainly represents the most frequently used subtraction technique in NLO calculations. It was first proposed within massless QCD by Catani and Seymour [5] and subsequently generalized to photon emission off massive fermions [6]<sup>1</sup> and to QCD with massive quarks [7,8]. Among the numerous applications of dipole subtraction, we merely mention the treatment of the electroweak corrections to  $e^+e^- \rightarrow 4$  fermions [11], which was the first complete treatment of a  $2 \rightarrow 4$  particle process at NLO. The formulation [5,7,8] of dipole subtraction for NLO QCD corrections assumes so-called infrared safety of observables, i.e. that all soft or collinear singularities cancel against their counterparts from the virtual corrections, either after parton-density redefi-

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<sup>1</sup>The case of light fermions, where masses appear as regulators, has also been worked out in Ref. [10].

nitions for initial-state singularities or due to the inclusiveness of event selection criteria in soft or collinear configurations for final-state singularities. In Ref. [6] the collinear singularities from photon radiation are regularized by physical fermion masses, and only for final-state radiation these were assumed to cancel due to inclusiveness of the observable.

In the following we generalize the method of Ref. [6] by dropping the latter assumption and by considering also other collinear-singular configurations involving photons, which are regularized by physical fermion masses:

1. In Section 2 we deal with *non-collinear-safe final-state radiation off light (anti-)fermions*  $f$ , where collinear singularities arise from the splitting  $f^* \rightarrow f\gamma$ . Here and in the following, asterisks indicate off-shell particles. By non-collinear-safe radiation we mean that a collinear fermion-photon system is not necessarily treated as one quasi-particle, which by contrast is the case in any collinear-safe observable. In collinear-safe situations, which are usually enforced by photon recombination or a jet algorithm, singularities from final-state radiation cancel according to the well-known Kinoshita–Lee–Nauenberg (KLN) theorem [12]. Non-collinear-safe final-state radiation off a fermion  $f$ , in general, leads to corrections  $\propto \alpha \ln m_f$  that are enhanced by a logarithm of a small fermion mass  $m_f$ .
2. Section 3 is devoted to *processes with incoming photons and outgoing light fermions*. Here the collinear-singular splitting is  $\gamma \rightarrow f\bar{f}^*$ , i.e. if an outgoing (anti-)fermion  $f$  is allowed to be scattered into the direction of the incoming photon, the cross section receives an enhancement  $\propto \ln m_f$  from this phase-space region.
3. In Section 4 we treat *processes with light fermion-antifermion pairs in the final state*, i.e. when an outgoing photon with low virtuality splits into an  $f\bar{f}$  pair,  $\gamma^* \rightarrow f\bar{f}$ . If the collinearly produced  $f\bar{f}$  pair can be distinguished from a plainly emitted photon (that has not split), the considered cross section again receives an enhancement  $\propto \ln m_f$ .
4. Finally, in Section 5 we concentrate on *processes with forward-scattered light (anti-)fermions*, where the splitting  $f \rightarrow f\gamma^*$  leads to a collinear singularity if the emitted photon is almost real. Again this phase-space region enhances the cross section by a factor  $\propto \ln m_f$ .

While Section 2 builds on the conventions and results of Ref. [6], Sections 3, 4, and 5 are self-contained and can be read independently.

Of course, the considered situations could all be treated by fully including a non-zero fermion mass  $m_f$  in the calculation. However, if  $m_f$  is small compared to typical scales in the process, which is the case for electrons or muons in almost all present and future high-energy collider experiments, such a procedure is very inconvenient. The presence of very small or large scale ratios jeopardizes the numerical stability of phase-space integrations, and mass terms significantly slow down the evaluation of matrix elements. The subtraction technique described in the following avoids these problems by completely isolating all mass singularities from squared matrix elements, so that finally only amplitudes for a massless fermion  $f$  are needed. We support particle polarization whenever relevant, in particular for all incoming particles. In order to facilitate cross-checks in applications, the corresponding formulas for the phase-space slicing approach are also provided.

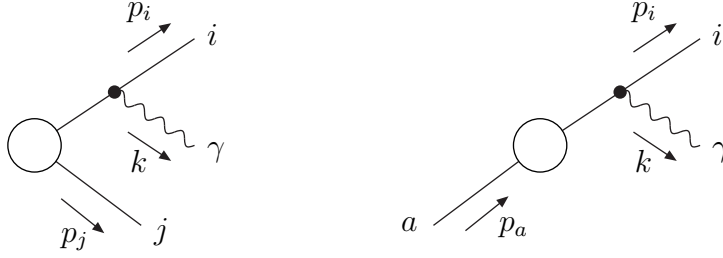


Figure 1: Generic diagrams for photonic final-state radiation off an emitter  $i$  with a spectator  $j$  or  $a$  in the final or initial state, respectively.

In Section 6 we demonstrate the use and the performance of the methods presented in Sections 3, 4, and 5 in the example  $e^- \gamma \rightarrow e^- \mu^- \mu^+$ . A summary is given in Section 7, and the appendices provide more details on and generalizations of the formulas presented in the main text. In particular, the derivation of the factorization formulas for processes with incoming polarized photons splitting into light fermions and for the forward scattering of incoming polarized light fermions is described there.

## 2 Non-collinear-safe photon radiation off final-state fermions

### 2.1 Dipole subtraction and non-collinear-safe observables

For any subtraction formalism the schematic form of the subtraction procedure to integrate the squared matrix element  $\sum_{\lambda_\gamma} |\mathcal{M}_1|^2$  (summed over photon polarizations  $\lambda_\gamma$ ) for real photon radiation over the  $(N+1)$ -particle phase space  $d\Phi_1$  reads

$$\int d\Phi_1 \sum_{\lambda_\gamma} |\mathcal{M}_1|^2 = \int d\Phi_1 \left( \sum_{\lambda_\gamma} |\mathcal{M}_1|^2 - |\mathcal{M}_{\text{sub}}|^2 \right) + \int d\tilde{\Phi}_0 \otimes \left( \int [dk] |\mathcal{M}_{\text{sub}}|^2 \right), \quad (2.1)$$

where  $d\tilde{\Phi}_0$  is a phase-space element of the corresponding non-radiative process and  $[dk]$  includes the photonic phase space that leads to the soft and collinear singularities. The two contributions involving the subtraction function  $|\mathcal{M}_{\text{sub}}|^2$  have to cancel each other, however, they will be evaluated separately. The subtraction function is constructed in such a way that the difference  $\sum_{\lambda_\gamma} |\mathcal{M}_1|^2 - |\mathcal{M}_{\text{sub}}|^2$  can be safely integrated over  $d\Phi_1$  numerically and that the singular integration of  $|\mathcal{M}_{\text{sub}}|^2$  over  $[dk]$  can be carried out analytically, followed by a safe numerical integration over  $d\tilde{\Phi}_0$ .

In the dipole subtraction formalism for photon radiation, the subtraction function is given by [6]

$$|\mathcal{M}_{\text{sub}}(\Phi_1; \kappa_f)|^2 = - \sum_{f \neq f'} Q_f \sigma_f Q_{f'} \sigma_{f'} e^2 g_{ff', \tau}^{(\text{sub})}(p_f, p_{f'}, k) \left| \mathcal{M}_0 \left( \tilde{\Phi}_{0, ff'}; \tau \kappa_f \right) \right|^2, \quad (2.2)$$

where the sum runs over all emitter-spectator pairs  $ff'$ , which are called dipoles. For a final-state emitter (final-state radiation), the two possible dipoles are illustrated in Fig. 1. The relative charges are denoted  $Q_f, Q_{f'}$ , and the sign factors  $\sigma_f, \sigma_{f'} = \pm 1$  correspond to the charge flow ( $\sigma_f = +1$  for incoming fermions and outgoing antifermions,  $\sigma_f = -1$

for outgoing fermions and incoming antifermions). The implicitly assumed summation over  $\tau = \pm$  accounts for a possible flip in the helicity of the emitter  $f$ , where  $\kappa_f = \pm$  is the sign of the helicity of  $f$  both in  $|\mathcal{M}_1|^2$  and  $|\mathcal{M}_{\text{sub}}|^2$ . The singular behaviour of the subtraction function is contained in the radiator functions  $g_{ff',\tau}^{(\text{sub})}(p_f, p_{f'}, k)$ , which depend on the emitter, spectator, and photon momenta  $p_f$ ,  $p_{f'}$ , and  $k$ , respectively. The squared lowest-order matrix element  $|\mathcal{M}_0|^2$  of the corresponding non-radiative process enters the subtraction function with modified emitter and spectator momenta  $\tilde{p}_f^{(ff')}$  and  $\tilde{p}_{f'}^{(ff')}$ . For a final-state emitter  $f$ , the momenta are related by  $p_f + k \pm p_{f'} = \tilde{p}_f^{(ff')} \pm \tilde{p}_{f'}^{(ff')}$ , where  $\pm$  refers to a spectator  $f'$  in the final or initial state, and the same set  $\{k_n\}$  of remaining particle momenta enters  $|\mathcal{M}_1|^2$  and  $|\mathcal{M}_0|^2$ . The modified momenta are constructed in such a way that  $\tilde{p}_f^{(ff')} \rightarrow p_f + k$  in the collinear limit ( $p_f k \rightarrow 0$ ).

Note that no collinear singularity exists for truly massive radiating particles  $f$ , because the invariant  $p_f k$  does not tend to zero if the photon emission angle becomes small (for fixed photon energy  $k^0$ ). In such cases the corresponding masses are kept non-zero in all amplitudes, in the subtraction functions, and in the kinematics, and the subtraction procedure works without problems. Collinear (or mass) singularities result if the mass  $m_f$  of a radiating particle is much smaller than the typical scale in the process under consideration. In such cases it is desirable to set  $m_f$  to zero whenever possible. In a subtraction technique this means that  $m_f = 0$  can be consistently used in the integral  $\int d\Phi_1 (\sum_{\lambda_\gamma} |\mathcal{M}_1|^2 - |\mathcal{M}_{\text{sub}}|^2)$ , but that the readded contribution  $\int [dk] |\mathcal{M}_{\text{sub}}|^2$  contains mass-singular terms of the form  $\alpha \ln m_f$ . If such mass singularities from collinear photon radiation do not completely cancel against their counterparts in the virtual corrections, the corresponding observable is *not collinear safe*. The dipole subtraction formalism as described in Ref. [6] is formulated to cover possible mass singularities from initial-state radiation, but assumes collinear safety w.r.t. final-state radiation.

In *collinear-safe* observables (w.r.t. final-state radiation), and only those are considered for light fermions in Ref. [6], a collinear fermion–photon system is treated as one quasi-particle, i.e., in the limit where  $f$  and  $\gamma$  become collinear only the sum  $p_f + k$  enters the procedures of implementing phase-space selection cuts or of sorting an event into a histogram bin of a differential distribution. Technically this level of inclusiveness is reached by *photon recombination*, a procedure that assigns the photon to the nearest charged particle if it is close enough to it. Of course, different variants for such an algorithm are possible, similar to jet algorithms in QCD. The recombination guarantees that for each photon radiation cone around a charged particle  $f$  the energy fraction

$$z_f = \frac{p_f^0}{p_f^0 + k^0} \quad (2.3)$$

is fully integrated over. According to the KLN theorem, no mass singularity connected with final-state radiation remains. Collinear safety facilitates the actual application of the subtraction procedure as indicated in Eq. (2.1). In this case the events resulting from the contributions of  $|\mathcal{M}_{\text{sub}}|^2$  can be consistently regarded as  $N$ -particle final states of the non-radiative process with particle momenta as going into  $|\mathcal{M}_0(\tilde{\Phi}_{0,ff'})|^2$ , i.e. the emitter and spectator momenta are given by  $\tilde{p}_f^{(ff')}$ ,  $\tilde{p}_{f'}^{(ff')}$ , respectively. Owing to  $\tilde{p}_f^{(ff')} \rightarrow p_f + k$  in the collinear limits, the difference  $\sum_{\lambda_\gamma} |\mathcal{M}_1|^2 - |\mathcal{M}_{\text{sub}}|^2$  can be integrated over all collinear

regions, because all events that differ only in the value of  $z_f$  enter cuts or histograms in the same way. The implicit *full* integration over all  $z_f$  in the collinear cones, on the other hand, implies that in the analytical integration of  $|\mathcal{M}_{\text{sub}}|^2$  over  $[dk]$  the  $z_f$  integrations can be carried out over the whole  $z_f$  range.

In *non-collinear-safe* observables (w.r.t. final-state radiation), not all photons within arbitrarily narrow collinear cones around outgoing charged particles are treated inclusively. For a fixed cone axis the integration over the corresponding variable  $z_f$  is constrained by a phase-space cut or by the boundary of a histogram bin. Consequently, mass-singular contributions of the form  $\alpha \ln m_f$  remain in the integral. Technically this means that the information on the variables  $z_f$  has to be exploited in the subtraction procedure of Eq. (2.1). The variables that take over the role of  $z_f$  in the individual dipole contributions in  $|\mathcal{M}_{\text{sub}}|^2$  are called  $z_{ij}$  and  $z_{ia}$  in Ref. [6], where  $f = i$  is a final-state emitter and  $j/a$  a final-/initial-state spectator. In the collinear limit they behave as  $z_{ij} \rightarrow z_i$  and  $z_{ia} \rightarrow z_i$ . Thus, the integral  $\int d\Phi_1 \left( \sum_{\lambda_\gamma} |\mathcal{M}_1|^2 - |\mathcal{M}_{\text{sub}}|^2 \right)$  can be performed over the whole phase space if the events associated with  $|\mathcal{M}_{\text{sub}}|^2$  are treated as  $(N+1)$ -particle event with momenta  $p_f \rightarrow z_{ff'} \tilde{p}_f^{(ff')}$ ,  $p_{f'} \rightarrow \tilde{p}_{f'}^{(ff')}$ , and  $k \rightarrow (1 - z_{ff'}) \tilde{p}_f^{(ff')}$ . This can be formalized by introducing a step function  $\Theta_{\text{cut}}(p_f, k, p_{f'}, \{k_n\})$  on the  $(N+1)$ -particle phase space which is 1 if the event passes the cuts and 0 otherwise. The set  $\{k_n\}$  simply contains the momenta of the remaining particles in the process. Making the dependence on  $\Theta_{\text{cut}}$  explicit, the first term on the r.h.s. of Eq. (2.1) reads

$$\int d\Phi_1 \left[ \sum_{\lambda_\gamma} |\mathcal{M}_1|^2 \Theta_{\text{cut}}(p_f, k, p_{f'}, \{k_n\}) - \sum_{f \neq f'} |\mathcal{M}_{\text{sub}, ff'}|^2 \Theta_{\text{cut}}(z_{ff'} \tilde{p}_f^{(ff')}, (1 - z_{ff'}) \tilde{p}_f^{(ff')}, \tilde{p}_{f'}^{(ff')}, \{k_n\}) \right], \quad (2.4)$$

where we have decomposed the subtraction function  $|\mathcal{M}_{\text{sub}}|^2$  into its subcontributions  $|\mathcal{M}_{\text{sub}, ff'}|^2$  of specific emitter-spectator pairs  $ff'$ . Apart from this refinement of the cut prescription in the subtraction part for non-collinear-safe observables, no modification in  $|\mathcal{M}_{\text{sub}}|^2$  is needed. Since its construction exactly proceeds as described in Sections 3 and 4 of Ref. [6], we do not repeat the individual steps in this paper.

However, the modification of the cut procedure requires a generalization of the evaluation of the second subtraction term on the r.h.s. of Eq. (2.1), because now the integral over  $z_{ff'}$  implicitly contained in  $[dk]$  depends on the cuts that define the observable. In the following two sections we work out the form of the necessary modifications, where we set up the formalism in such a way that it reduces to the procedure described in Ref. [6] for a collinear-safe situation, while the non-collinear-safe case is covered upon including extra contributions.

## 2.2 Final-state emitter and final-state spectator

For a final-state emitter  $i$  and a final-state spectator  $j$  with masses  $m_i$  and  $m_j$  the integral of  $g_{ij, \tau}^{(\text{sub})}(p_i, p_j, k)$  over  $[dk]$  is proportional to

$$G_{ij, \tau}^{(\text{sub})}(P_{ij}^2) = \frac{\bar{P}_{ij}^4}{2\sqrt{\lambda_{ij}}} \int_{y_1}^{y_2} dy_{ij} (1 - y_{ij}) \int_{z_1(y_{ij})}^{z_2(y_{ij})} dz_{ij} g_{ij, \tau}^{(\text{sub})}(p_i, p_j, k), \quad (2.5)$$

where the definitions of Sections 3.1 and 4.1 of Ref. [6] are used. There the results for  $G_{ij,\tau}^{(\text{sub})}(P_{ij}^2)$  with generic or light masses are given in Eqs. (4.10) and (3.7), respectively. In order to leave the integration over  $z_{ij}$  open, the order of the two integrations has to be interchanged, and the integral solely taken over  $y_{ij}$  is needed. Therefore, we define

$$\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z_{ij}) = \frac{\bar{P}_{ij}^4}{2\sqrt{\lambda_{ij}}} \int_{y_1(z_{ij})}^{y_2(z_{ij})} dy_{ij} (1 - y_{ij}) g_{ij,\tau}^{(\text{sub})}(p_i, p_j, k). \quad (2.6)$$

Note that no finite photon mass  $m_\gamma$  is needed in the function  $\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z)$  in practice, because the soft singularity appearing at  $z \rightarrow 1$  can be split off by employing a  $[\dots]_+$  prescription in the variable  $z$ ,

$$\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z) = G_{ij,\tau}^{(\text{sub})}(P_{ij}^2) \delta(1 - z) + [\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z)]_+. \quad (2.7)$$

This procedure shifts the soft singularity into the quantity  $G_{ij,\tau}^{(\text{sub})}(P_{ij}^2)$ , which is already known from Ref. [6]. Moreover, the generalization to non-collinear-safe integrals simply reduces to the extra term  $[\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z)]_+$ , which cancels out for collinear-safe integrals where the full  $z$ -integration is carried out.

For arbitrary values of  $m_i$  and  $m_j$  a compact analytical result of  $\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z)$  cannot be achieved because of the complicated structure of the integration boundary. Note, however, that only the limit  $m_i \rightarrow 0$  of a light emitter is relevant, since for truly massive emitters no mass singularity results. The case of a massive spectator  $j$  is presented in App. A; here we restrict ourselves to the simpler but important special case  $m_j = 0$ .

In the limit  $m_i \rightarrow 0$  and  $m_j = m_\gamma = 0$  the boundary of the  $y_{ij}$  integration is asymptotically given by

$$y_1(z) = \frac{m_i^2(1 - z)}{P_{ij}^2 z}, \quad y_2(z) = 1, \quad (2.8)$$

and the functions and quantities relevant in the integrand  $g_{ij,\tau}^{(\text{sub})}$  behave as

$$p_i k = \frac{P_{ij}^2}{2} y_{ij}, \quad R_{ij}(y) = 1 - y, \quad r_{ij}(y) = 1. \quad (2.9)$$

The evaluation of Eq. (2.6) becomes very simple and yields

$$\begin{aligned} \bar{\mathcal{G}}_{ij,+}^{(\text{sub})}(P_{ij}^2, z) &= P_{ff}(z) \left[ \ln\left(\frac{P_{ij}^2 z}{m_i^2}\right) - 1 \right] + (1 + z) \ln(1 - z), \\ \bar{\mathcal{G}}_{ij,-}^{(\text{sub})}(P_{ij}^2, z) &= 1 - z, \end{aligned} \quad (2.10)$$

where  $P_{ff}(z)$  is the splitting function,

$$P_{ff}(z) = \frac{1 + z^2}{1 - z}. \quad (2.11)$$

Equation (2.10) is correct up to terms suppressed by factors of  $m_i$ . For completeness, we repeat the form of the full integral  $G_{ij,\tau}^{(\text{sub})}(P_{ij}^2)$  in the case of light masses,

$$G_{ij,+}^{(\text{sub})}(P_{ij}^2) = \mathcal{L}(P_{ij}^2, m_i^2) - \frac{\pi^2}{3} + 1, \quad G_{ij,-}^{(\text{sub})}(P_{ij}^2) = \frac{1}{2}, \quad (2.12)$$

with the auxiliary function

$$\mathcal{L}(P^2, m^2) = \ln\left(\frac{m^2}{P^2}\right) \ln\left(\frac{m_\gamma^2}{P^2}\right) + \ln\left(\frac{m_\gamma^2}{P^2}\right) - \frac{1}{2} \ln^2\left(\frac{m^2}{P^2}\right) + \frac{1}{2} \ln\left(\frac{m^2}{P^2}\right), \quad (2.13)$$

which are taken from Eqs. (3.7) and (3.8) of Ref. [6].<sup>2</sup>

Finally, we give the explicit form of the  $ij$  contribution  $|\mathcal{M}_{\text{sub},ij}(\Phi_1)|^2$  to the phase-space integral of the subtraction function,

$$\begin{aligned} \int d\Phi_1 |\mathcal{M}_{\text{sub},ij}(\Phi_1; \kappa_i)|^2 &= -\frac{\alpha}{2\pi} Q_i \sigma_i Q_j \sigma_j \int d\tilde{\Phi}_{0,ij} \int_0^1 dz \\ &\times \left\{ G_{ij,\tau}^{(\text{sub})}(P_{ij}^2) \delta(1-z) + [\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z)]_+ \right\} \\ &\times |\mathcal{M}_0(\tilde{p}_i, \tilde{p}_j; \tau \kappa_i)|^2 \Theta_{\text{cut}}(p_i = z\tilde{p}_i, k = (1-z)\tilde{p}_i, \tilde{p}_j, \{k_n\}), \end{aligned} \quad (2.14)$$

generalizing Eq. (3.6) of Ref. [6]. While  $\tilde{p}_i, \tilde{p}_j, \{k_n\}$  are the momenta corresponding to the generated phase-space point in  $\tilde{\Phi}_{0,ij}$ , the momenta  $p_i$  and  $k$  result from  $\tilde{p}_i$  via a simple rescaling with the independently generated variable  $z$ . The invariant  $P_{ij}^2$  is calculated via  $P_{ij}^2 = (\tilde{p}_i + \tilde{p}_j)^2$  independently of  $z$ . The arguments of the step function  $\Theta_{\text{cut}}(p_i, k, \tilde{p}_j, \{k_n\})$  indicate on which momenta phase-space cuts are imposed.

For unpolarized fermions the results of this section have already been described in Ref. [13], where electroweak radiative corrections to the processes  $\gamma\gamma \rightarrow \text{WW} \rightarrow 4$  fermions were calculated. In this calculation the results for non-collinear-safe differential cross sections were also cross-checked against results obtained with phase-space slicing. Another comparison between the described subtraction procedure and phase-space slicing has been performed in the calculation of electroweak corrections to the Higgs decay processes  $\text{H} \rightarrow \text{WW}/\text{ZZ} \rightarrow 4$  fermions [14].

### 2.3 Final-state emitter and initial-state spectator

For the treatment of a final-state emitter  $i$  and an initial-state spectator  $a$ , we consistently make use of the definitions of Sections 3.2 and 4.2 of Ref. [6]. In this paper we only consider light particles in the initial state, because the masses of incoming particles are much smaller than the scattering energies at almost all present and future colliders. Therefore, the spectator mass  $m_a$  can be set to zero from the beginning, which simplifies the formulas considerably.

Before we consider the non-collinear-safe situation, we briefly repeat the concept of the collinear-safe case described in Ref. [6]. Following Eqs. (4.24) and (4.27) from there, the inclusive integral of  $g_{ia,\tau}^{(\text{sub})}(p_i, p_a, k)$  over  $[dk]$  is proportional to

$$G_{ia,\tau}^{(\text{sub})}(P_{ia}^2) = \int_0^{x_1} dx \mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x) \quad (2.15)$$

with

$$\mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x_{ia}) = -\frac{\bar{P}_{ia}^2}{2} \int_{z_1(x_{ia})}^{z_2(x_{ia})} dz_{ia} g_{ia,\tau}^{(\text{sub})}(p_i, p_a, k), \quad (2.16)$$

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<sup>2</sup>If dimensional regularization is used to regularize the soft singularity instead of a finite photon mass, the photon-mass logarithm in  $\mathcal{L}$  has to be replaced according to  $\ln(m_\gamma^2) \rightarrow (4\pi\mu^2)^\epsilon \Gamma(1+\epsilon)/\epsilon + \mathcal{O}(\epsilon)$ , where  $D = 4 - 2\epsilon$  is the dimension and  $\mu$  the reference mass of dimensional regularization.



where we could set the lower limit  $x_0$  of the  $x_{ia}$ -integration to zero because of  $m_a = 0$ . Since, however, the squared lowest-order matrix element  $|\mathcal{M}_0|^2$  multiplying  $g_{ia,\tau}^{(\text{sub})}$  in Eq. (2.2) depends on the variable  $x_{ia}$ , the integration of  $|\mathcal{M}_{\text{sub}}|^2$  over  $x = x_{ia}$  is performed employing a  $[\dots]_+$  prescription,

$$\begin{aligned} & -\frac{\bar{P}_{ia}^2}{2} \int_0^{x_1} dx_{ia} \int_{z_1(x_{ia})}^{z_2(x_{ia})} dz_{ia} g_{ia,\tau}^{(\text{sub})}(p_i, p_a, k) \cdots \\ & = \int_0^1 dx \left\{ G_{ia,\tau}^{(\text{sub})}(P_{ia}^2) \delta(1-x) + [\mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x)]_+ \right\} \cdots \end{aligned} \quad (2.17)$$

This integration, where the ellipses stand for  $x$ -dependent functions such as the squared lowest-order matrix elements and flux factors, is usually done numerically. Since the soft and collinear singularities occur at  $x \rightarrow x_1 = 1 - \mathcal{O}(m_\gamma)$ , the singular parts are entirely contained in  $G_{ia,\tau}^{(\text{sub})}(P_{ia}^2)$  in Eq. (2.17), and the upper limit  $x_1$  could be replaced by 1 in the actual  $x$ -integration. For completeness we give the explicit form of the functions  $G_{ia,\tau}^{(\text{sub})}$  and  $\mathcal{G}_{ia,\tau}^{(\text{sub})}$  in the limit  $m_i \rightarrow 0$ ,

$$\begin{aligned} G_{ia,+}^{(\text{sub})}(P_{ia}^2) &= \mathcal{L}(|P_{ia}^2|, m_i^2) - \frac{\pi^2}{2} + 1, & G_{ia,-}^{(\text{sub})}(P_{ia}^2) &= \frac{1}{2}, \\ \mathcal{G}_{ia,+}^{(\text{sub})}(P_{ia}^2, x) &= \frac{1}{1-x} \left[ 2 \ln\left(\frac{2-x}{1-x}\right) - \frac{3}{2} \right], & \mathcal{G}_{ia,-}^{(\text{sub})}(P_{ia}^2, x) &= 0, \end{aligned} \quad (2.18)$$

which are taken from Eqs. (3.19) and (3.20) of Ref. [6].

In a non-collinear-safe situation, the ellipses on the l.h.s. of Eq. (2.17) also involve  $z_{ia}$ -dependent functions, as e.g.  $\theta$ -functions for cuts or event selection. Thus, also the integration over  $z_{ia}$  has to be performed numerically in this case, and we have to generalize Eq. (2.17) in an appropriate way. To this end, we generalize the usual  $[\dots]_+$  prescription in the following way. Writing

$$\int d^n \mathbf{r} [g(\mathbf{r})]_{+, (a)}^{(r_i)} f(\mathbf{r}) \equiv \int d^n \mathbf{r} g(\mathbf{r}) \left( f(\mathbf{r}) - f(\mathbf{r})|_{r_i=a} \right) \quad (2.19)$$

for the  $[\dots]_+$  prescription in the  $r_i$ -integration in a multiple integral over  $n$  variables  $r_k$  ( $k = 1, \dots, n$ ), we can iterate this definition to two-dimensional integrals according to

$$\begin{aligned} \int d^n \mathbf{r} [g(\mathbf{r})]_{+, (a,b)}^{(r_i, r_j)} f(\mathbf{r}) &\equiv \int d^n \mathbf{r} \left[ [g(\mathbf{r})]_{+, (a)}^{(r_i)} \right]_{+, (b)}^{(r_j)} f(\mathbf{r}) \\ &= \int d^n \mathbf{r} g(\mathbf{r}) \left( f(\mathbf{r}) - f(\mathbf{r})|_{r_i=a} - f(\mathbf{r})|_{r_j=b} + f(\mathbf{r})|_{\substack{r_i=a \\ r_j=b}} \right). \end{aligned} \quad (2.20)$$

In the notation  $[g(\mathbf{r})]_{+, (a)}^{(r_i)}$  we omit the superscript  $(r_i)$  if  $g(\mathbf{r})$  depends only on the integration variable  $r_i$ , and we omit the subscripts  $(a)$  or  $(a, b)$  if  $a = 1$  or  $a = b = 1$ . This obviously recovers the usual notation for the one-dimensional prescription used above. Introducing a double  $[\dots]_+$  prescription in  $x = x_{ia}$  and  $z = z_{ia}$ , we generalize Eq. (2.17) to

$$\begin{aligned} & -\frac{\bar{P}_{ia}^2}{2} \int_0^{x_1} dx \int_{z_1(x)}^{z_2(x)} dz g_{ia,\tau}^{(\text{sub})}(p_i, p_a, k) \cdots \\ & = \int_0^1 dx \int_0^1 dz \left\{ G_{ia,\tau}^{(\text{sub})}(P_{ia}^2) \delta(1-x) \delta(1-z) + [\mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x)]_+ \delta(1-z) \right. \end{aligned}$$

$$+ \left[ \bar{\mathcal{G}}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, z) \right]_+ \delta(1-x) + \left[ \bar{g}_{ia,\tau}^{(\text{sub})}(x, z) \right]_+^{(x,z)} \} \dots \quad (2.21)$$

If the functions hidden in the ellipses do not depend on  $z$ , the last two terms within the curly brackets do not contribute and the formula reduces to Eq. (2.17).

We derive Eq. (2.21) and the explicit form of the two extra terms in two steps. In the derivation we quantify the previous ellipses by the regular test function  $f(x, z)$ . The first step introduces a  $[\dots]_+$  prescription in the  $x$ -integration of the l.h.s. of Eq. (2.21) after interchanging the order of the integrations,

$$\begin{aligned} I[f] &\equiv -\frac{\bar{P}_{ia}^2}{2} \int_0^{x_1} dx \int_{z_1(x)}^{z_2(x)} dz g_{ia,\tau}^{(\text{sub})} f(x, z) \\ &= -\frac{\bar{P}_{ia}^2}{2} \int_0^1 dz \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})} f(x, z) \\ &= -\frac{\bar{P}_{ia}^2}{2} \int_0^1 dz \int_0^{x_1(z)} dx \left\{ \left[ g_{ia,\tau}^{(\text{sub})} \right]_+^{(x)} f(x, z) + g_{ia,\tau}^{(\text{sub})} f(x_1(z), z) \right\}. \end{aligned} \quad (2.22)$$

The upper limit  $x_1(z)$  of the  $x$ -integration follows upon solving the explicit form of the limits  $z_{1,2}(x)$  (given in Eq. (4.22) of Ref. [6]) for  $x$ . The full form of  $x_1(z)$  is rather complicated for finite  $m_\gamma$ , but in the following it is only needed for  $m_\gamma = 0$ , where it simplifies to

$$x_1(z) \Big|_{m_\gamma=0} = \frac{\bar{P}_{ia}^2 z}{\bar{P}_{ia}^2 z - m_i^2(1-z)}. \quad (2.23)$$

Note that soft or collinear singularities result from the region of highest  $x$  values,  $x \rightarrow x_1 = \max\{x_1(z)\}$ , so that the first term in curly brackets in Eq. (2.22) is free of such singularities owing to the  $[\dots]_+$  regularization. Thus, we can set  $m_i \rightarrow 0$  in this part, i.e. in particular  $x_1(z) \rightarrow 1$ , yielding

$$I[f] = -\frac{\bar{P}_{ia}^2}{2} \int_0^1 dz \left\{ \int_0^1 dx \left[ g_{ia,\tau}^{(\text{sub})} \right]_+^{(x)} f(x, z) + f(x_1(z), z) \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})} \right\}. \quad (2.24)$$

In the second step we introduce a  $[\dots]_+$  prescription for the  $z$ -integration in both terms,

$$\begin{aligned} I[f] &= -\frac{\bar{P}_{ia}^2}{2} \int_0^1 dz \left\{ \int_0^1 dx \left[ \left[ g_{ia,\tau}^{(\text{sub})} \right]_+^{(x)} \right]_+^{(z)} f(x, z) + \int_0^1 dx \left[ g_{ia,\tau}^{(\text{sub})} \right]_+^{(x)} f(x, 1) \right. \\ &\quad \left. + f(x_1(z), z) \left[ \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})} \right]_+^{(z)} + f(x_1(1), 1) \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})} \right\} \\ &= -\frac{\bar{P}_{ia}^2}{2} \int_0^1 dx \int_0^1 dz \left[ g_{ia,\tau}^{(\text{sub})} \right]_+^{(x,z)} f(x, z) - \frac{\bar{P}_{ia}^2}{2} \int_0^1 dx f(x, 1) \left[ \int_0^1 dz g_{ia,\tau}^{(\text{sub})} \right]_+^{(x)} \\ &\quad - \frac{\bar{P}_{ia}^2}{2} \int_0^1 dz f(x_1(z), z) \left[ \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})} \right]_+^{(z)} - \frac{\bar{P}_{ia}^2}{2} f(x_1(1), 1) \int_0^1 dz \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})}. \end{aligned} \quad (2.25)$$

In the second equality we just reordered some factors and integrations. Since all integrals over the test function  $f$  are now free of singularities, i.e. the singularities are contained

in the integrals multiplying  $f$ , we can set the regulator masses  $m_\gamma$  and  $m_i$  to zero in the arguments of  $f$ . Thus, we can write

$$I[f] = \int_0^1 dx \int_0^1 dz \left[ \bar{g}_{ia,\tau}^{(\text{sub})}(x, z) \right]_+^{(x,z)} f(x, z) + \int_0^1 dx f(x, 1) \left[ \mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x) \right]_+ \\ + \int_0^1 dz f(1, z) \left[ \bar{\mathcal{G}}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, z) \right]_+ + f(1, 1) G_{ia,\tau}^{(\text{sub})}(P_{ia}^2) \quad (2.26)$$

with the abbreviations

$$\bar{g}_{ia,\tau}^{(\text{sub})}(x, z) = -\frac{\bar{P}_{ia}^2}{2} g_{ia,\tau}^{(\text{sub})} \Big|_{\substack{m_\gamma=0 \\ m_i=0}}, \\ \mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x) = -\frac{\bar{P}_{ia}^2}{2} \int_0^1 dz g_{ia,\tau}^{(\text{sub})} \Big|_{\substack{m_\gamma=0 \\ m_i=0}}, \\ \bar{\mathcal{G}}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, z) = -\frac{\bar{P}_{ia}^2}{2} \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})} \Big|_{m_\gamma=0}, \\ G_{ia,\tau}^{(\text{sub})}(P_{ia}^2) = -\frac{\bar{P}_{ia}^2}{2} \int_0^1 dz \int_0^{x_1(z)} dx g_{ia,\tau}^{(\text{sub})}. \quad (2.27)$$

Equation (2.26) is equivalent to the anticipated result (2.21), which was to be shown. The explicit results for  $\mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x)$  and  $G_{ia,\tau}^{(\text{sub})}(P_{ia}^2)$  have already been given above in Eq. (2.18), the two remaining functions are easily evaluated to

$$\bar{g}_{ia,+}^{(\text{sub})}(x, z) = \frac{1}{1-x} \left( \frac{2}{2-x-z} - 1 - z \right), \quad \bar{g}_{ia,-}^{(\text{sub})}(x, z) = 0, \\ \bar{\mathcal{G}}_{ia,+}^{(\text{sub})}(P_{ia}^2, z) = P_{ff}(z) \left[ \ln \left( \frac{-P_{ia}^2 z}{m_i^2} \right) - 1 \right] - \frac{2 \ln(2-z)}{1-z} + (1+z) \ln(1-z), \\ \bar{\mathcal{G}}_{ia,-}^{(\text{sub})}(P_{ia}^2, z) = 1 - z. \quad (2.28)$$

The collinear singularity  $\propto \ln m_i$  that appears in non-collinear-safe observables is contained in the function  $\bar{\mathcal{G}}_{ia,+}^{(\text{sub})}(P_{ia}^2, z)$ .

The resulting  $ia$  contribution  $|\mathcal{M}_{\text{sub},ia}(\Phi_1)|^2$  to the phase-space integral of the subtraction function reads

$$\int d\Phi_1 |\mathcal{M}_{\text{sub},ia}(\Phi_1; \kappa_i)|^2 = -\frac{\alpha}{2\pi} Q_a \sigma_a Q_i \sigma_i \int_0^1 dx \int d\tilde{\Phi}_{0,ia}(P_{ia}^2, x) \int_0^1 dz \\ \times \Theta_{\text{cut}}(p_i = z\tilde{p}_i(x), k = (1-z)\tilde{p}_i(x), \{\tilde{k}_n(x)\}) \\ \times \frac{1}{x} \left\{ G_{ia,\tau}^{(\text{sub})}(P_{ia}^2) \delta(1-x) \delta(1-z) + \left[ \mathcal{G}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, x) \right]_+ \delta(1-z) \right. \\ \left. + \left[ \bar{\mathcal{G}}_{ia,\tau}^{(\text{sub})}(P_{ia}^2, z) \right]_+ \delta(1-x) + \left[ \bar{g}_{ia,\tau}^{(\text{sub})}(x, z) \right]_+^{(x,z)} \right\} |\mathcal{M}_0(\tilde{p}_i(x), \tilde{p}_a(x); \tau \kappa_i)|^2, \quad (2.29)$$

which generalizes Eq. (3.18) of Ref. [6]. Again, the arguments of the step function  $\Theta_{\text{cut}}(p_i, k, \{\tilde{k}_n\})$  indicate on which momenta phase-space cuts are imposed. We recall that  $\tilde{\Phi}_{0,ia}$  is the phase space of momenta  $\tilde{p}_i(x)$  and  $\{\tilde{k}_n(x)\}$  (without final-state radiation) with rescaled incoming momentum  $\tilde{p}_a(x) = x p_a$  instead of the original incoming

momentum  $p_a$ . In the actual evaluation of Eq. (2.29), thus, the two phase-space points  $\tilde{\Phi}_{0,ia}(P_{ia}^2, x)$  and  $\tilde{\Phi}_{0,ia}(P_{ia}^2, x = 1)$  have to be generated for each value of  $x$  owing to the plus prescription in  $x$ . The relevant value of the invariant  $P_{ia}^2$  is then calculated separately via  $P_{ia}^2 = (\tilde{p}_i - \tilde{p}_a)^2$  for each of the two points, so that  $P_{ia}^2$  results from the momenta entering the matrix element  $\mathcal{M}_0$  in both cases.<sup>3</sup> The variable  $z$ , however, is generated independently of the phase-space points and does not influence the kinematics in the matrix element.

The combination of the subtraction procedures described in this and the previous section has been successfully applied and compared to results obtained with phase-space slicing in the calculations of electroweak corrections to Drell–Yan-like W-boson production,  $pp \rightarrow W \rightarrow \nu_l l + X$ , and to deep-inelastic neutrino scattering,  $\nu_\mu N \rightarrow \nu_\mu/\mu + X$ , building on the calculations discussed in Refs. [15,16] and [17], respectively.

## 2.4 Phase-space slicing

In the phase-space slicing approach the soft and collinear phase-space regions are excluded in the (numerical) integration of the squared amplitude of the real-emission process. In the so-called two-cutoff slicing method the soft region is cut off by demanding that the photon energy  $k^0$  should be larger than a lower cut  $\Delta E$  which is much smaller than any relevant energy scale of the process. The collinear regions are excluded by demanding that each angle of the photon with any other direction of a light charged particle should be larger than the cut value  $\Delta\theta \ll 1$ . Note that this phase-space splitting is not Lorentz invariant. In the soft and collinear regions the photon phase space can be integrated out analytically by employing the general factorization properties of the squared amplitudes, which are, e.g., discussed in Section 2.2 of Ref. [6] (including polarization effects). General results for the integral over the soft region can, e.g., be found in Refs. [18,19]. The integrals over the collinear regions for final-state radiation can be easily obtained from intermediate results of the two previous sections as follows.

The cuts defining the collinear region for the photon–emitter system of Section 2.2 translate into new limits for the integration variables  $y_{ij}$  and  $z_{ij}$ ,

$$\frac{m_i^2(1 - z_{ij})}{\bar{P}_{ij}^2 z_{ij}} < y_{ij} < \frac{(p_i^0)^2}{\bar{P}_{ij}^2} \frac{1 - z_{ij}}{z_{ij}} \Delta\theta^2, \quad 0 < z_{ij} < 1 - \frac{\Delta E}{p_i^0}, \quad (2.30)$$

which are asymptotically valid up to the relevant order in  $m_i \rightarrow 0$ . With these new limits on  $y_{ij}$  we evaluate the integral defined in Eq. (2.6) and obtain

$$\bar{\mathcal{G}}_+^{(\text{sli})}(p_i^0, z) = P_{ff}(z) \left[ \ln \left( \frac{(p_i^0)^2 \Delta\theta^2}{m_i^2} \right) - 1 \right], \quad \bar{\mathcal{G}}_-^{(\text{sli})}(p_i^0, z) = 1 - z. \quad (2.31)$$

The integrals of these functions over  $z = z_{ij}$  are given by

$$G_+^{(\text{sli})}(p_i^0) = - \left[ \ln \left( \frac{\Delta E^2}{(p_i^0)^2} \right) + \frac{3}{2} \right] \left[ \ln \left( \frac{(p_i^0)^2 \Delta\theta^2}{m_i^2} \right) - 1 \right], \quad G_-^{(\text{sli})}(p_i^0) = \frac{1}{2}. \quad (2.32)$$

---

<sup>3</sup>For a more formal explanation of this subtle but important point we refer to the discussion at the end of Section 6.3 of Ref. [8].

As it should be, in these results the dependence on the spectator particle  $j$  completely disappears, because it was only needed in the phase-space parametrization. We also note that the same results can be obtained from Section 2.3, where the limits on  $x_{ia}$  and  $z_{ia}$  are changed to

$$\frac{m_i^2(1-z_{ia})}{-\bar{P}_{ia}^2 z_{ia} + m_i^2(1-z_{ia})} < 1 - x_{ia} < \frac{(p_i^0)^2}{-\bar{P}_{ia}^2} \frac{1-z_{ia}}{z_{ia}} \Delta\theta^2, \quad 0 < z_{ia} < 1 - \frac{\Delta E}{p_i^0}. \quad (2.33)$$

Using the functions  $\bar{\mathcal{G}}_\tau^{(\text{sli})}$  and  $G_\tau^{(\text{sli})}$ , the integral over the collinear photon emission cone around particle  $i$  reads

$$\begin{aligned} \int_{\text{coll},i} d\Phi_1 |\mathcal{M}_1(\Phi_1; \kappa_i)|^2 &= \frac{\alpha}{2\pi} Q_i^2 \int d\tilde{\Phi}_0 \int_0^1 dz \left\{ G_\tau^{(\text{sli})}(p_i^0) \delta(1-z) + [\bar{\mathcal{G}}_\tau^{(\text{sli})}(p_i^0, z)]_+ \right\} \\ &\quad \times |\mathcal{M}_0(\tilde{p}_i; \tau \kappa_i)|^2 \Theta_{\text{cut}}(p_i = z\tilde{p}_i, k = (1-z)\tilde{p}_i, \{k_n\}), \end{aligned} \quad (2.34)$$

where the momenta  $\tilde{p}_i$  and  $\{k_n\}$  belong to the phase-space point  $\tilde{\Phi}_0$ . Of course, apart from the polarization issue this is a well-known result which can be found in various papers [3].<sup>4</sup>

### 3 Collinear singularities from $\gamma \rightarrow f\bar{f}^*$ splittings

#### 3.1 Asymptotics in the collinear limit

We consider a generic scattering process

$$\gamma(k, \lambda_\gamma) + a(p_a) \rightarrow f(p_f) + X, \quad (3.1)$$

where the momenta of the particles are indicated in parentheses and  $\lambda_\gamma = \pm$  is the photon helicity. Here  $a$  is any massless incoming particle and  $f$  is an outgoing light fermion or antifermion. The remainder  $X$  may contain additional light fermions which can be treated in the same way as  $f$ . For later use, we define the squared centre-of-mass energy  $s$ ,

$$s = (p_a + k)^2 = 2p_a k. \quad (3.2)$$

The collinear singularity in the squared matrix element  $|\mathcal{M}_{\gamma a \rightarrow f X}|^2$  occurs if the angle  $\theta_f$  between  $f$  and the incoming  $\gamma$  becomes small; in this limit the scalar product  $(kp_f)$  is of  $\mathcal{O}(m_f^2)$ , where  $m_f$  is the small mass of  $f$ . Neglecting terms that are irrelevant in the limit  $m_f \rightarrow 0$  the squared matrix element  $|\mathcal{M}_{\gamma a \rightarrow f X}(k, p_a, p_f; \lambda_\gamma)|^2$  for a definite photon helicity  $\lambda_\gamma = \pm$  (but summed over the polarizations of  $f$ ) asymptotically behaves like

$$|\mathcal{M}_{\gamma a \rightarrow f X}(k, p_a, p_f; \lambda_\gamma)|^2 \underset{kp_f \rightarrow 0}{\sim} Q_f^2 e^2 h_\tau^{\gamma f}(k, p_f) |\mathcal{M}_{\bar{f} a \rightarrow X}(p_{\bar{f}} = xk, p_a; \kappa_{\bar{f}} = \tau \lambda_\gamma)|^2, \quad (3.3)$$

where  $x = 1 - p_f^0/k^0$  and  $Q_f e$  is the electric charge of  $f$ . The matrix element  $\mathcal{M}_{\bar{f} a \rightarrow X}$  corresponds to the related process  $\bar{f}a \rightarrow X$  that results from  $\gamma a (\rightarrow f\bar{f}^* a) \rightarrow fX$  upon cutting the  $\bar{f}^*$  line in all diagrams involving the splitting  $\gamma \rightarrow f\bar{f}^*$  (see also Fig. 2). The incoming momenta relevant in the different matrix elements are given in parentheses.

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<sup>4</sup>Descriptions of phase-space slicing for initial-state radiation off unpolarized particles can also be found in Ref. [3]; the case of polarized incoming particles is, e.g., treated in Ref. [20].

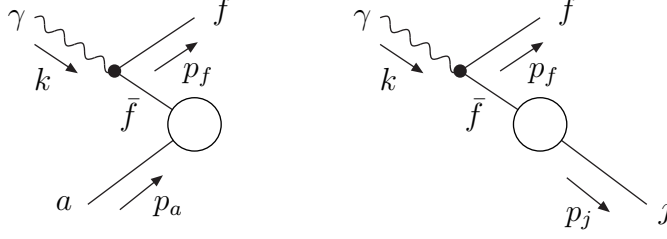


Figure 2: Generic diagrams for the splittings  $\gamma \rightarrow f \bar{f}^*$  with an initial-state spectator  $a$  or a final-state spectator  $j$ .

Moreover, in Eq. (3.3) we assume a summation over  $\tau = \pm$ , where  $\tau = \pm$  refers to the two cases where the sign  $\kappa_{\bar{f}}$  of the  $\bar{f}$  helicity is equal or opposite to the photon helicity  $\lambda_\gamma$ . The functions  $h_\tau^{\gamma f}(k, p_f)$ , which rule the structure of the collinear singularity, are given by

$$\begin{aligned} h_+^{\gamma f}(k, p_f) &= \frac{1}{x(kp_f)} \left( P_{f\gamma}(x) + \frac{xm_f^2}{kp_f} \right) - h_-^{\gamma f}(k, p_f), \\ h_-^{\gamma f}(k, p_f) &= \frac{1}{x(kp_f)} (1-x) \left( 1 - x - \frac{m_f^2}{2kp_f} \right), \end{aligned} \quad (3.4)$$

with the splitting function

$$P_{f\gamma}(x) = (1-x)^2 + x^2. \quad (3.5)$$

The derivation of this result is given in App. B.1.

Note that the collinear singularity for  $kp_f \rightarrow 0$  can be attributed to a single external leg (namely  $\bar{f}$ ) of the related hard process  $fa \rightarrow X$ . Thus, there is no need to construct the subtraction function  $|\mathcal{M}_{\text{sub}}|^2$  from several dipole contributions  $\propto Q_f Q_{f'}$ . Instead we can construct  $|\mathcal{M}_{\text{sub}}|^2$  as a single term  $\propto Q_f^2$ . Nevertheless we select a spectator  $f'$  to the emitter  $f$  for the phase-space construction, which proceeds in complete analogy to the photon radiation case. We have the freedom to choose any particle in the initial or final state as spectator. In the following we describe the “dipole” formalism in two variants: one with a spectator from the initial state, another with a spectator from the final state. The two situations are illustrated in Fig. 2.

### 3.2 Initial-state spectator

The function that is subtracted from the integrand  $|\mathcal{M}_{\gamma a \rightarrow fX}(k, p_a, p_f; \lambda_\gamma)|^2$  is defined as follows,

$$|\mathcal{M}_{\text{sub}}(\lambda_\gamma)|^2 = Q_f^2 e^2 h_\tau^{\gamma f, a}(k, p_f, p_a) \left| \mathcal{M}_{\bar{f}a \rightarrow X}(\tilde{p}_{\bar{f}}, p_a, \{\tilde{k}_n\}; \kappa_{\bar{f}} = \tau \lambda_\gamma) \right|^2, \quad (3.6)$$

with the radiator functions

$$\begin{aligned} h_+^{\gamma f, a}(k, p_f, p_a) &= \frac{1}{x_{f, \gamma a}(kp_f)} \left( P_{f\gamma}(x_{f, \gamma a}) + \frac{x_{f, \gamma a} m_f^2}{kp_f} \right) - h_-^{\gamma f, a}(k, p_f, p_a), \\ h_-^{\gamma f, a}(k, p_f, p_a) &= \frac{1}{x_{f, \gamma a}(kp_f)} (1 - x_{f, \gamma a}) \left( 1 - x_{f, \gamma a} - \frac{m_f^2}{2kp_f} \right), \end{aligned} \quad (3.7)$$

and the auxiliary quantity

$$x_{f,\gamma a} = \frac{p_a k - p_f k - p_a p_f}{p_a k}. \quad (3.8)$$

Here we kept the dependence on a finite  $m_f$ , because it is needed in the analytical treatment of the singular phase-space integration below. The modified momenta  $\tilde{p}_{\tilde{f}}$  and  $\{\tilde{k}_n\}$  entering the squared matrix element on the r.h.s. of Eq. (3.6) will only be needed for  $m_f = 0$  in applications with small values of  $m_f$ . In this limit they can be chosen as

$$\tilde{p}_{\tilde{f}}^\mu(x) = x k^\mu, \quad \tilde{p}_f^\mu = \tilde{p}_{\tilde{f}}^\mu(x_{f,\gamma a}), \quad \tilde{k}_n^\mu = \Lambda^\mu{}_\nu k_n^\nu \quad (3.9)$$

with the Lorentz transformation matrix  $\Lambda^\mu{}_\nu$  given by

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu - \frac{(P + \tilde{P})^\mu (P + \tilde{P})_\nu}{P^2 + P\tilde{P}} + \frac{2\tilde{P}^\mu P_\nu}{P^2}, \quad (3.10)$$

$$P^\mu = p_a^\mu + k^\mu - p_f^\mu, \quad \tilde{P}^\mu(x) = p_a^\mu + \tilde{p}_{\tilde{f}}^\mu(x), \quad \tilde{P}^\mu = p_a^\mu + \tilde{p}_{\tilde{f}}^\mu. \quad (3.11)$$

It is straightforward to check that  $|\mathcal{M}_{\text{sub}}|^2$  possesses the same asymptotic behaviour as  $|\mathcal{M}_{\gamma a \rightarrow fX}|^2$  in Eq. (3.3) in the collinear limit with  $m_f \rightarrow 0$ . Thus, the difference  $|\mathcal{M}_{\gamma a \rightarrow fX}|^2 - |\mathcal{M}_{\text{sub}}|^2$  can be integrated numerically for  $m_f = 0$ .

The correct dependence of  $|\mathcal{M}_{\text{sub}}|^2$  (and the related kinematics) on a finite  $m_f$  is, however, needed when this function is integrated over  $\theta_f$  leading to the collinear singularity for  $\theta_f \rightarrow 0$ . The actual analytical integration can be done as described in Ref. [6] (even for finite  $m_a$  and  $m_f$ ). Here we only sketch the individual steps and give the final result. The  $(N + 1)$ -particle phase space is first split into the corresponding  $N$ -particle phase space and the integral over the remaining degrees of freedom that contain the singularity,

$$\int d\phi(p_f, P; k + p_a) = \int_0^{x_1} dx \int d\phi(\tilde{P}(x); \tilde{p}_{\tilde{f}}(x) + p_a) \int [dp_f(s, x, y_{f,\gamma a})], \quad (3.12)$$

with the explicit parametrization

$$\int [dp_f(s, x, y_{f,\gamma a})] = \frac{s}{4(2\pi)^3} \int_{y_1(x)}^{y_2(x)} dy_{f,\gamma a} \int d\phi_f. \quad (3.13)$$

The upper kinematical limit of the parameter  $x = x_{f,\gamma a}$  is given by

$$x_1 = 1 - \frac{2m_f}{\sqrt{s}}, \quad (3.14)$$

but in the limit  $m_f \rightarrow 0$  we can set  $x_1 = 1$ . While the integration of the azimuthal angle  $\phi_f$  of  $f$  simply yields a factor  $2\pi$ , the integration over the auxiliary parameter

$$y_{f,\gamma a} = \frac{kp_f}{kp_a} = \frac{2kp_f}{s} \quad (3.15)$$

with the boundary

$$y_{1,2}(x) = \frac{1}{2} \left[ 1 - x \mp \sqrt{(1 - x)^2 - \frac{4m_f^2}{s}} \right] \quad (3.16)$$

is less trivial. Defining

$$\mathcal{H}_\tau^{\gamma f, a}(s, x) = \frac{xs}{2} \int_{y_1(x)}^{y_2(x)} dy_{f, \gamma a} h_\tau^{\gamma f, a}(k, p_f, p_a), \quad (3.17)$$

the result of this straightforward integration (for  $m_f \rightarrow 0$ ) is

$$\begin{aligned} \mathcal{H}_+^{\gamma f, a}(s, x) &= P_{f\gamma}(x) \ln\left(\frac{s(1-x)^2}{m_f^2}\right) + 2x(1-x) - \mathcal{H}_-^{\gamma f, a}(s, x), \\ \mathcal{H}_-^{\gamma f, a}(s, x) &= (1-x)^2 \ln\left(\frac{s(1-x)^2}{m_f^2}\right) - (1-x)^2. \end{aligned} \quad (3.18)$$

For clarity we finally give the contribution  $\sigma_{\gamma a \rightarrow fX}^{\text{sub}}$  that has to be added to the result for the cross section obtained from the integral of the difference  $|\mathcal{M}_{\gamma a \rightarrow fX}|^2 - |\mathcal{M}_{\text{sub}}|^2$ ,

$$\sigma_{\gamma a \rightarrow fX}^{\text{sub}}(k, p_a; \lambda_\gamma) = N_{c,f} \frac{Q_f^2 \alpha}{2\pi} \int_0^1 dx \mathcal{H}_\tau^{\gamma f, a}(s, x) \sigma_{\bar{f}a \rightarrow X}(p_{\bar{f}} = xk, p_a; \kappa_{\bar{f}} = \tau \lambda_\gamma). \quad (3.19)$$

Although formulated for integrated cross sections, the previous formula can be used to calculate any differential cross section after obvious modifications.

For the case of unpolarized photons this subtraction variant has already been briefly described in Ref. [17], where it was applied to the contributions to deep-inelastic neutrino scattering,  $\nu_\mu N \rightarrow \nu_\mu/\mu + X$ , that are induced by a photon distribution function of the nucleon  $N$ . Moreover, the method presented here was successfully used in the calculation of photon-induced real corrections to Drell–Yan-like  $W$  production (see Section 10 of Ref. [1] and Ref. [16]) and of photon- and gluon-induced real corrections to Higgs production via vector-boson fusion at the LHC [21]. All these results were also cross-checked against phase-space slicing.

### 3.3 Final-state spectator

As an alternative to the case of an initial-state spectator described in the previous section, we here present the treatment with a possibly massive final-state spectator  $j$  with mass  $m_j$ , i.e. we consider the process

$$\gamma(k) + a(p_a) \rightarrow f(p_f) + j(p_j) + X. \quad (3.20)$$

The initial-state particle  $a$  is assumed massless in the following, but all formulas can be generalized to  $m_a \neq 0$  following closely the treatment of phase space described in Section 4.2 of Ref. [6]. The subtraction function now is constructed as follows,

$$|\mathcal{M}_{\text{sub}}(\lambda_\gamma)|^2 = Q_f^2 e^2 h_{j,\tau}^{\gamma f}(k, p_f, p_j) |\mathcal{M}_{\bar{f}a \rightarrow jX}(\tilde{p}_{\bar{f}}, p_a, \tilde{p}_j; \kappa_{\bar{f}} = \tau \lambda_\gamma)|^2, \quad (3.21)$$

with the radiator functions

$$\begin{aligned} h_{j,+}^{\gamma f}(k, p_f, p_j) &= \frac{1}{x_{fj,\gamma}(kp_f)} \left( P_{f\gamma}(x_{fj,\gamma}) + \frac{x_{fj,\gamma} m_f^2}{kp_f} \right) - h_{j,-}^{\gamma f}(k, p_f, p_j), \\ h_{j,-}^{\gamma f}(k, p_f, p_j) &= \frac{1}{x_{fj,\gamma}(kp_f)} (1 - x_{fj,\gamma}) \left( 1 - x_{fj,\gamma} - \frac{m_f^2}{2kp_f} \right) \end{aligned} \quad (3.22)$$



and the auxiliary parameter

$$x_{fj,\gamma} = \frac{kp_j + kp_f - p_f p_j}{kp_j + kp_f}. \quad (3.23)$$

The momenta  $\tilde{p}_{\bar{f}}$  and  $\tilde{p}_j$  are given by

$$\tilde{p}_{\bar{f}}^\mu(x) = xk^\mu, \quad \tilde{p}_{\bar{f}}^\mu = \tilde{p}_{\bar{f}}^\mu(x_{fj,\gamma}), \quad \tilde{p}_j^\mu = P^\mu + \tilde{p}_{\bar{f}}^\mu, \quad P^\mu = p_f^\mu + p_j^\mu - k^\mu, \quad (3.24)$$

while the momenta of the other particles are unaffected. Note that this construction of momenta is based on the restriction  $m_f = 0$ , which is used in the integration of the difference  $|\mathcal{M}_{\gamma a \rightarrow fjX}|^2 - |\mathcal{M}_{\text{sub}}|^2$ .

In the integration of  $|\mathcal{M}_{\text{sub}}|^2$  over the collinear-singular phase space, of course, the correct dependence on a finite  $m_f$  is required. Owing to the finite spectator mass  $m_j$ , this procedure is quite involved; we sketch it in App. B.2. Here we only present the results needed in practice. The cross-section contribution  $\sigma_{\gamma a \rightarrow fjX}^{\text{sub}}$  that has to be added to the integrated difference  $|\mathcal{M}_{\gamma a \rightarrow fjX}|^2 - |\mathcal{M}_{\text{sub}}|^2$  is given by

$$\sigma_{\gamma a \rightarrow fjX}^{\text{sub}}(k, p_a; \lambda_\gamma) = N_{c,f} \frac{Q_f^2 \alpha}{2\pi} \int_0^1 dx \mathcal{H}_{j,\tau}^{\gamma f}(P^2, x) \sigma_{\bar{f} a \rightarrow jX}(p_{\bar{f}} = xk, p_a; \kappa_{\bar{f}} = \tau \lambda_\gamma), \quad (3.25)$$

where the collinear singularity is again contained in the kernels

$$\begin{aligned} \mathcal{H}_{j,+}^{\gamma f}(P^2, x) &= -P_{f\gamma}(x) \ln \left[ \frac{m_j^2 x}{(m_j^2 - P^2)(1-x)} \left( 1 + \frac{m_j^2 x}{(m_j^2 - P^2)(1-x)} \right) \right] + 2x(1-x) \\ &\quad - \mathcal{H}_{j,-}^{\gamma f}(P^2, x), \\ \mathcal{H}_{j,-}^{\gamma f}(P^2, x) &= -(1-x)^2 \ln \left[ \frac{m_j^2 x}{(m_j^2 - P^2)(1-x)} \left( 1 + \frac{m_j^2 x}{(m_j^2 - P^2)(1-x)} \right) \right] - (1-x)^2. \end{aligned} \quad (3.26)$$

Of course, the singular contributions  $\propto \ln m_f$  have the same form as in the case of an initial-state spectator discussed in the previous section.

### 3.4 Phase-space slicing

From the results of the two previous sections, the corresponding formulas for the phase-space slicing approach can be easily obtained. The collinear region, which is omitted in the phase-space integration, is defined by the restriction  $\theta_f < \Delta\theta$  on the fermion emission angle  $\theta_f$  in some given reference frame.

In Section 3.2 this constraint translates into new limits on the variable  $y_{f,\gamma a}$ ,

$$\frac{m_f^2}{s(1-x_{f,\gamma a})} < y_{f,\gamma a} < \frac{(k^0)^2(1-x_{f,\gamma a})}{s} \Delta\theta^2, \quad (3.27)$$

which modifies the result of the integral analogously defined to Eq. (3.17) to

$$\begin{aligned} \mathcal{H}_+^{\gamma f}(k^0, x) &= P_{f\gamma}(x) \ln \left( \frac{(k^0)^2(1-x)^2 \Delta\theta^2}{m_f^2} \right) + 2x(1-x) - \mathcal{H}_-^{\gamma f}(k^0, x), \\ \mathcal{H}_-^{\gamma f}(k^0, x) &= (1-x)^2 \ln \left( \frac{(k^0)^2(1-x)^2 \Delta\theta^2}{m_f^2} \right) - (1-x)^2. \end{aligned} \quad (3.28)$$

The cross-section contribution of the collinear region of  $f$  then reads

$$\sigma_{\gamma a \rightarrow fX}^{\text{coll},f}(k, p_a; \lambda_\gamma) = N_{c,f} \frac{Q_f^2 \alpha}{2\pi} \int_0^1 dx \mathcal{H}_\tau^{\gamma f}(k^0, x) \sigma_{\bar{f}a \rightarrow X}(p_{\bar{f}} = xk, p_a; \kappa_{\bar{f}} = \tau \lambda_\gamma). \quad (3.29)$$

The same result is obtained from Section 3.3 with App. B.2, where the new limits on  $z_{fj,\gamma}$  read

$$\frac{m_f^2 x_{fj,\gamma}}{-\bar{P}^2(1 - x_{fj,\gamma})} < 1 - z_{fj,\gamma} < \frac{(k^0)^2 x_{fj,\gamma}(1 - x_{fj,\gamma})}{-\bar{P}^2} \Delta\theta^2. \quad (3.30)$$

## 4 Collinear singularities from $\gamma^* \rightarrow f\bar{f}$ splittings

### 4.1 Asymptotics in the collinear limit

We consider a generic scattering process

$$a(p_a) + b(p_b) \rightarrow f(p_f) + \bar{f}(p_{\bar{f}}) + X, \quad (4.1)$$

where the momenta of the particles are indicated in parentheses. Depending on the particle content of the remainder  $X$ , there may be additional, independent collinear-singular configurations, but we are interested in the region where the invariant mass  $(p_f + p_{\bar{f}})^2 = 2m_f^2 + 2p_f p_{\bar{f}}$  of the produced fermion–antifermion pair  $f\bar{f}$  becomes of the order  $\mathcal{O}(m_f^2)$ , where  $m_f$  is small compared to typical scales in the process. The singular behaviour of the full squared matrix element  $|\mathcal{M}_{ab \rightarrow f\bar{f}X}(p_f, p_{\bar{f}})|^2$  entirely originates from diagrams containing a  $\gamma^* \rightarrow f\bar{f}$  splitting, i.e. the singularity is related to the subprocess  $ab \rightarrow \gamma X$ . For the matrix element of this subprocess we write  $\mathcal{M}_{ab \rightarrow \gamma X} = T_{ab \rightarrow \gamma X}^\mu(\tilde{k}) \varepsilon_{\lambda_\gamma, \mu}(\tilde{k})^*$ , where  $T_{ab \rightarrow \gamma X}^\mu(\tilde{k})$  is the amplitude without the photon polarization vector  $\varepsilon_{\lambda_\gamma, \mu}(\tilde{k})^*$ . In the collinear limit  $p_f p_{\bar{f}} \rightarrow 0$  the light-like momentum  $\tilde{k}$  is equal to  $k = p_f + p_{\bar{f}}$  up to mass-suppressed terms. Neglecting terms that are irrelevant in the limit  $m_f \rightarrow 0$  the squared matrix element  $|\mathcal{M}_{ab \rightarrow f\bar{f}X}(p_f, p_{\bar{f}})|^2$  asymptotically behaves like

$$|\mathcal{M}_{ab \rightarrow f\bar{f}X}(p_f, p_{\bar{f}})|^2 \underset{p_f p_{\bar{f}} \rightarrow 0}{\sim} N_{c,f} Q_f^2 e^2 h_{f\bar{f}, \mu\nu}(p_f, p_{\bar{f}}) T_{ab \rightarrow \gamma X}^\mu(\tilde{k})^* T_{ab \rightarrow \gamma X}^\nu(\tilde{k}), \quad (4.2)$$

where

$$h_{f\bar{f}, \mu\nu}(p_f, p_{\bar{f}}) = \frac{2}{(p_f + p_{\bar{f}})^2} \left[ -g_{\mu\nu} + 4z(1 - z) \frac{k_{\perp, \mu} k_{\perp, \nu}}{k_{\perp}^2 - m_f^2} \right], \quad z = \frac{p_f^0}{k^0}, \quad (4.3)$$

and  $N_{c,f}$  is the colour multiplicity of  $f$  ( $N_{c,\text{lepton}} = 1$ ,  $N_{c,\text{quark}} = 3$ ). The momentum  $k_{\perp}$  is the component of  $p_f$  that is orthogonal to the collinear axis defined by  $k$ , i.e.  $kk_{\perp} = 0$ , and becomes of  $\mathcal{O}(m_f)$  in the collinear limit. An explicit prescription for the construction of  $k_{\perp}$  can, e.g., be found in Ref. [8], where the analogous case of the gluonic splitting into massive quarks  $Q, g^* \rightarrow Q\bar{Q}$ , is worked out. It is important to realize that  $h_{f\bar{f}, \mu\nu}$  in Eq. (4.2) is not proportional to the polarization sum  $E_{\mu\nu} = \sum_{\lambda_\gamma} \varepsilon_{\lambda_\gamma, \mu}(\tilde{k})^* \varepsilon_{\lambda_\gamma, \nu}(\tilde{k})$  of the photon, so that the r.h.s. is not proportional to the polarization-summed squared amplitude  $|\mathcal{M}_{ab \rightarrow \gamma X}|^2$  of the subprocess. This spin correlation has to be taken care of in the construction of an appropriate subtraction function in order to guarantee a point-wise cancellation of the singular behaviour in the collinear phase-space region. The spin

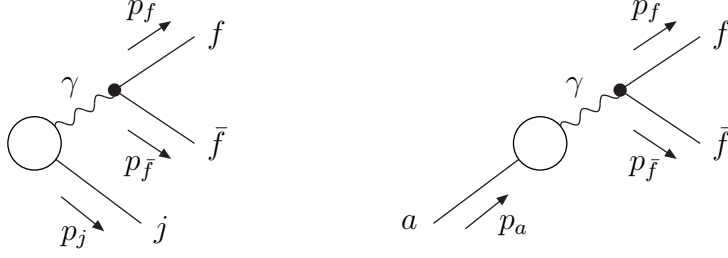


Figure 3: Generic diagrams for the splittings  $\gamma^* \rightarrow f\bar{f}$  with an initial-state spectator  $a$  or a final-state spectator  $j$ , where  $f$  is a light fermion or antifermion.

correlation encoded in  $h_{f\bar{f},\mu\nu}$  drops out if the average over the azimuthal angle  $\phi_f$  of the  $\gamma^* \rightarrow f\bar{f}$  splitting plane around the collinear axis is taken.<sup>5</sup> Indicating this averaging by  $\langle \dots \rangle_{\phi_f} \equiv \int d\phi_f / (2\pi)$ , we get  $\langle h_{f\bar{f},\mu\nu} \rangle_{\phi_f} = E_{\mu\nu} h_{f\bar{f}}$  with (in four space-time dimensions)

$$h_{f\bar{f}}(p_f, p_{\bar{f}}) = \frac{2}{(p_f + p_{\bar{f}})^2} \left[ P_{f\gamma}(z) + \frac{2m_f^2}{(p_f + p_{\bar{f}})^2} \right] \quad (4.4)$$

up to terms that are further suppressed by factors of  $m_f$ . The averaged squared matrix element behaves as

$$\langle |\mathcal{M}_{ab \rightarrow f\bar{f}X}(p_f, p_{\bar{f}})|^2 \rangle_{\phi_f} \xrightarrow{p_f p_{\bar{f}} \rightarrow 0} N_{c,f} Q_f^2 e^2 h_{f\bar{f}}(p_f, p_{\bar{f}}) |\mathcal{M}_{ab \rightarrow \gamma X}(\tilde{k})|^2. \quad (4.5)$$

Since the collinear singularity for  $p_f p_{\bar{f}} \rightarrow 0$  can be attributed to a single external leg (the photon) of the related hard process  $ab \rightarrow \gamma X$ , also in this case there is no need to construct the subtraction function  $|\mathcal{M}_{\text{sub}}|^2$  from several dipole contributions. The function  $|\mathcal{M}_{\text{sub}}|^2$  can be chosen as a single term  $\propto Q_f^2$ . Nevertheless a spectator is selected for the phase-space construction, as in the previous section. In the following we describe the “dipole” construction in two variants: one with a spectator from the initial state, another with a spectator from the final state. The two situations are illustrated in Fig. 3.

## 4.2 Initial-state spectator

We define the subtraction function as

$$|\mathcal{M}_{\text{sub}}|^2 = N_{c,f} Q_f^2 e^2 h_{f\bar{f},\mu\nu}^a(p_f, p_{\bar{f}}, p_a) T_{ab \rightarrow \gamma X}^\mu(\tilde{p}_a, \tilde{k})^* T_{ab \rightarrow \gamma X}^\nu(\tilde{p}_a, \tilde{k}) \quad (4.6)$$

with

$$h_{f\bar{f}}^{a,\mu\nu}(p_f, p_{\bar{f}}, p_a) = \frac{2}{(p_f + p_{\bar{f}})^2} \left[ -g^{\mu\nu} - \frac{4}{(p_f + p_{\bar{f}})^2} (z_{f\bar{f},a} p_f^\mu - \bar{z}_{f\bar{f},a} p_{\bar{f}}^\mu) (z_{f\bar{f},a} p_f^\nu - \bar{z}_{f\bar{f},a} p_{\bar{f}}^\nu) \right] \quad (4.7)$$

<sup>5</sup>As described in Refs. [5,8], for unpolarized situations this average can be easily obtained upon contraction with the projector  $\frac{1}{2}d_{\mu\nu}(k) = \frac{1}{2}[-g_{\mu\nu} + (\text{“gauge terms” involving } k^\mu \text{ or } k^\nu)]/(1-\epsilon)$ , which fulfills  $-g^{\mu\nu}d_{\mu\nu}(k) = 2(1-\epsilon)$  and  $k^\mu d_{\mu\nu}(k) = 0$  in  $D = 4 - 2\epsilon$  space-time dimensions.

and the auxiliary parameters

$$x_{f\bar{f},a} = \frac{p_a p_f + p_a p_{\bar{f}} - p_f p_{\bar{f}} - m_f^2}{p_a p_f + p_a p_{\bar{f}}}, \quad z_{f\bar{f},a} = 1 - \bar{z}_{f\bar{f},a} = \frac{p_a p_f}{p_a p_f + p_a p_{\bar{f}}}. \quad (4.8)$$

The auxiliary momenta entering the amplitude for the related process  $ab \rightarrow \gamma X$  are given by

$$\begin{aligned} \tilde{p}_a^\mu(x) &= x p_a^\mu, & \tilde{p}_a^\mu &= \tilde{p}_a^\mu(x_{f\bar{f},a}), \\ \tilde{k}^\mu(x) &= P^\mu + \tilde{p}_a^\mu(x), & \tilde{k}^\mu &= \tilde{k}^\mu(x_{f\bar{f},a}), & P^\mu &= p_f^\mu + p_{\bar{f}}^\mu - p_a^\mu, \end{aligned} \quad (4.9)$$

while the momenta of the other particles remain unchanged. In these equations we kept the dependence on  $m_f$ , but of course in the numerical integration of  $|\mathcal{M}_{ab \rightarrow f\bar{f}X}|^2 - |\mathcal{M}_{\text{sub}}|^2$  we can set  $m_f$  to zero, because we are only interested in the limit  $m_f \rightarrow 0$ . For the integration of  $|\mathcal{M}_{\text{sub}}|^2$  over the collinear-singular region, we need the  $m_f$ -dependence of the spin average of  $h_{f\bar{f}}^{a,\mu\nu}$ ,

$$h_{f\bar{f}}^a(p_f, p_{\bar{f}}, p_a) = \frac{2}{(p_f + p_{\bar{f}})^2} \left[ P_{f\gamma}(z_{f\bar{f},a}) + \frac{2m_f^2}{(p_f + p_{\bar{f}})^2} \right], \quad (4.10)$$

and an appropriate phase-space splitting,

$$\int d\phi(p_f, p_{\bar{f}}; P + p_a) = \int_0^{x_1} dx \int d\phi(\tilde{k}(x); P + \tilde{p}_a(x)) \int [dp_f(P^2, x, z)], \quad (4.11)$$

where we have used the shorthands  $x = x_{f\bar{f},a}$  and  $z = z_{f\bar{f},a}$ . The explicit form of  $\int [dp_f]$  reads

$$\int [dp_f(P^2, x, z)] = \frac{-P^2}{4(2\pi)^3} \frac{1}{x} \int_{z_1(x)}^{z_2(x)} dz \int d\phi_f \quad (4.12)$$

with the integration limits for the variables  $x$  and  $z$

$$x_1 = \frac{P^2}{P^2 - 4m_f^2}, \quad z_{1,2}(x) = \frac{1}{2} \left( 1 \pm \sqrt{\frac{x_1 - x}{x_1(1 - x)}} \right). \quad (4.13)$$

Separating the singular contributions as described in Section 2.3, we rewrite the integral of  $h_{f\bar{f}}^a$  for  $m_f \rightarrow 0$  as

$$\begin{aligned} & -\frac{P^2}{2} \int_0^{x_1} dx \int_{z_1(x)}^{z_2(x)} dz h_{f\bar{f}}^a(p_f, p_{\bar{f}}, p_a) \cdots \\ &= \int_0^1 dx \int_0^1 dz \left\{ H_{f\bar{f}}^a(P^2) \delta(1-x) \delta(1-z) + [\mathcal{H}_{f\bar{f}}^a(P^2, x)]_+ \delta(1-z) \right. \\ & \quad \left. + [\bar{\mathcal{H}}_{f\bar{f}}^a(P^2, z)]_+ \delta(1-x) + [\bar{h}_{f\bar{f}}^a(x, z)]_+^{(x,z)} \right\} \cdots. \end{aligned} \quad (4.14)$$

The new functions  $\bar{h}_{f\bar{f}}^a$ , etc., defined here are obtained from obvious substitutions and straightforward integrations,

$$\bar{h}_{f\bar{f}}^a(x, z) = \frac{x}{1-x} P_{f\gamma}(z),$$

$$\begin{aligned}
\mathcal{H}_{f\bar{f}}^a(P^2, x) &= \frac{2x}{3(1-x)}, \\
\bar{\mathcal{H}}_{f\bar{f}}^a(P^2, z) &= P_{f\gamma}(z) \left[ \ln \left( \frac{-P^2 z(1-z)}{m_f^2} \right) - 1 \right] + 2z(1-z), \\
H_{f\bar{f}}^a(P^2) &= \frac{2}{3} \ln \left( \frac{-P^2}{m_f^2} \right) - \frac{16}{9}.
\end{aligned} \tag{4.15}$$

Using these functions the phase-space integral of the subtraction function reads

$$\begin{aligned}
\int d\Phi_{f\bar{f}} |\mathcal{M}_{\text{sub}}(\Phi_{f\bar{f}})|^2 &= N_{c,f} \frac{Q_f^2 \alpha}{2\pi} \int_0^1 dx \int d\tilde{\Phi}_\gamma(P^2, x) \int_0^1 dz \\
&\times \Theta_{\text{cut}}(p_f = z\tilde{k}(x), p_{\bar{f}} = (1-z)\tilde{k}(x), \{\tilde{k}_n(x)\}) \\
&\times \frac{1}{x} \left\{ H_{f\bar{f}}^a(P^2) \delta(1-x) \delta(1-z) + [\mathcal{H}_{f\bar{f}}^a(P^2, x)]_+ \delta(1-z) \right. \\
&\quad \left. + [\bar{\mathcal{H}}_{f\bar{f}}^a(P^2, z)]_+ \delta(1-x) + [\bar{h}_{f\bar{f}}^a(x, z)]_+^{(x,z)} \right\} |\mathcal{M}_{ab \rightarrow \gamma X}(\tilde{p}_a(x), \tilde{k}(x))|^2, \tag{4.16}
\end{aligned}$$

where we have made explicit which momenta enter the cut function  $\Theta_{\text{cut}}(p_f, p_{\bar{f}}, \{\tilde{k}_n\})$ . Concerning the phase-space integration over  $d\tilde{\Phi}_\gamma(P^2, x)$  and its integration over the boost parameter  $x$  the same comments as made after Eq. (2.29) apply. There are actually two phase-space points for each  $x$  value to be generated (one for  $x < 1$  and another for  $x = 1$ ), each determining momenta  $\tilde{p}_a(x), \tilde{k}(x), \{\tilde{k}_n(x)\}$  for the evaluation of  $P^2$  and the matrix elements. The generation of the parameter  $z$  proceeds independently, and the squared amplitude  $|\mathcal{M}_{ab \rightarrow \gamma X}|^2$  in Eq. (4.16) does not depend on  $z$ . Thus, if the full range in  $z$  is integrated over, i.e. if the collinear  $f\bar{f}$  pair is treated as a single quasiparticle in the cut procedure, the last two terms in curly brackets do not contribute. In this case the fermion-mass logarithm is entirely contained in the  $H_{f\bar{f}}^a$  contribution. According to the KLN theorem this contribution will be completely compensated by virtual  $\mathcal{O}(\alpha)$  corrections to the process  $ab \rightarrow \gamma X$  if collinear  $f\bar{f}$  pairs are not distinguished from emitted photons.

### 4.3 Final-state spectator

Since the case with a massive final-state spectator  $j$  is quite involved, we here present the formalism for  $m_j = 0$  and give the details for the massive case in App. C.

For  $m_f = m_j = 0$ , the subtraction function can be defined as

$$|\mathcal{M}_{\text{sub}}|^2 = N_{c,f} Q_f^2 e^2 h_{f\bar{f},j,\mu\nu}(p_f, p_{\bar{f}}, p_j) T_{ab \rightarrow \gamma j X}^\mu(\tilde{k}, \tilde{p}_j)^* T_{ab \rightarrow \gamma j X}^\nu(\tilde{k}, \tilde{p}_j) \tag{4.17}$$

with

$$h_{f\bar{f},j}^{\mu\nu}(p_f, p_{\bar{f}}, p_j) = \frac{2}{(p_f + p_{\bar{f}})^2} \left[ -g^{\mu\nu} - \frac{2}{p_f p_{\bar{f}}} (z_{f\bar{f}j} p_f^\mu - \bar{z}_{f\bar{f}j} p_{\bar{f}}^\mu) (z_{f\bar{f}j} p_f^\nu - \bar{z}_{f\bar{f}j} p_{\bar{f}}^\nu) \right] \tag{4.18}$$

and the auxiliary parameters

$$z_{f\bar{f}j} = 1 - \bar{z}_{f\bar{f}j} = \frac{p_f p_j}{p_f p_j + p_{\bar{f}} p_j}, \quad y_{f\bar{f}j} = \frac{p_f p_{\bar{f}}}{p_f p_j + p_{\bar{f}} p_j + p_f p_{\bar{f}}}. \tag{4.19}$$

The new momenta entering the amplitude for the related process  $ab \rightarrow \gamma j X$  are given by

$$\tilde{p}_j^\mu = p_j^\mu / (1 - y_{f\bar{f}j}), \quad \tilde{k}^\mu = P^\mu - \tilde{p}_j^\mu, \quad P^\mu = p_f^\mu + p_{\bar{f}}^\mu + p_j^\mu, \quad (4.20)$$

whereas all remaining momenta  $k_n$  of particles in  $X$  remain unchanged. Equation (4.17) can be used to integrate the difference  $|\mathcal{M}_{ab \rightarrow f\bar{f}jX}|^2 - |\mathcal{M}_{\text{sub}}|^2$  for massless fermions  $f$ . In order to integrate  $|\mathcal{M}_{\text{sub}}|^2$  over the collinear-singular region, the dependence on  $m_f$  has to be taken into account. Details of this procedure can be found in App. C. The result can be written in the form

$$\begin{aligned} \int d\Phi_{f\bar{f}} |\mathcal{M}_{\text{sub}}(\Phi_{f\bar{f}})|^2 &= N_{c,f} \frac{Q_f^2 \alpha}{2\pi} \int d\tilde{\Phi}_\gamma \int_0^1 dz \Theta_{\text{cut}}(p_f = z\tilde{k}, p_{\bar{f}} = (1-z)\tilde{k}, \tilde{p}_j, \{k_n\}) \\ &\quad \times \left\{ H_{f\bar{f},j}(P^2) \delta(1-z) + [\bar{\mathcal{H}}_{f\bar{f},j}(P^2, z)]_+ \right\} |\mathcal{M}_{ab \rightarrow \gamma j X}(\tilde{k}, \tilde{p}_j)|^2 \end{aligned} \quad (4.21)$$

with

$$\begin{aligned} \bar{\mathcal{H}}_{f\bar{f},j}(P^2, z) &= P_{f\gamma}(z) \left[ \ln\left(\frac{P^2 z(1-z)}{m_f^2}\right) - 1 \right] + 2z(1-z), \\ H_{f\bar{f},j}(P^2) &= \frac{2}{3} \ln\left(\frac{P^2}{m_f^2}\right) - \frac{16}{9}. \end{aligned} \quad (4.22)$$

The momenta  $\tilde{k}, \tilde{p}_j, \{k_n\}$  directly correspond to the generated phase-space point in  $\tilde{\Phi}_\gamma$ , while the parameter  $z$  is generated independently. The comments on the  $z$ -integration made at the end of the previous subsection apply also here. The squared amplitude  $|\mathcal{M}_{ab \rightarrow \gamma j X}|^2$  in Eq. (4.21) does not depend on  $z$ , and thus, if the event selection for  $f$  and  $\bar{f}$  is inclusive in the collinear region of the  $\gamma^* \rightarrow f\bar{f}$  splitting, the integral over  $z$  trivially reduces to the factor  $H_{f\bar{f},j}(P^2)$ .

#### 4.4 Phase-space slicing

Here we again deduce the integral over the collinear phase-space region which is needed in the slicing approach. This region can, e.g., be defined by restricting the angle  $\theta_{f\bar{f}}$  between the  $f$  and  $\bar{f}$  directions to small values,  $\theta_{f\bar{f}} < \Delta\theta \ll 1$ .

In Section 4.2 this restriction leads to new limits in  $x_{f\bar{f},a}$  and  $z_{f\bar{f},a}$ ,

$$\frac{m_f^2}{-P^2 z_{f\bar{f},a} (1 - z_{f\bar{f},a})} < 1 - x_{f\bar{f},a} < \frac{(k^0)^2}{-P^2} z_{f\bar{f},a} (1 - z_{f\bar{f},a}) \Delta\theta^2, \quad 0 < z_{f\bar{f},a} < 1, \quad (4.23)$$

where  $k^0 = p_f^0 + p_{\bar{f}}^0$  is the energy in the  $f\bar{f}$  system. This modifies the integrated results to

$$\begin{aligned} \bar{\mathcal{H}}_{f\bar{f}}(k^0, z) &= P_{f\gamma}(z) \ln\left(\frac{(k^0)^2 z^2 (1-z)^2 \Delta\theta^2}{m_f^2}\right) + 2z(1-z), \\ H_{f\bar{f}}(k^0) &= \frac{2}{3} \ln\left(\frac{(k^0)^2 \Delta\theta^2}{m_f^2}\right) - \frac{23}{9}, \end{aligned} \quad (4.24)$$

where  $\bar{\mathcal{H}}_{f\bar{f}}$  and  $H_{f\bar{f}}$  are defined analogously to Eq. (4.14). The integral of the squared matrix element over the collinear regions then reads

$$\begin{aligned}
\int_{\theta_{f\bar{f}} < \Delta\theta} d\Phi_{f\bar{f}} |\mathcal{M}_{ab \rightarrow f\bar{f}X}(\Phi_{f\bar{f}})|^2 &= N_{c,f} \frac{Q_f^2 \alpha}{2\pi} \int d\tilde{\Phi}_\gamma \int_0^1 dz \\
&\times \Theta_{\text{cut}}(p_f = z\tilde{k}, p_{\bar{f}} = (1-z)\tilde{k}, \{\tilde{k}_n\}) \\
&\times \left\{ H_{f\bar{f}}(k^0) \delta(1-z) + [\bar{\mathcal{H}}_{f\bar{f}}(k^0, z)]_+ \right\} |\mathcal{M}_{ab \rightarrow \gamma X}(\tilde{k})|^2.
\end{aligned} \tag{4.25}$$

The same results can be obtained from Section 4.3 with App. C, where the new limits on the integration variables are given by

$$\frac{m_f^2}{\bar{P}^2} \frac{z_{f\bar{f}j}^2 + (1 - z_{f\bar{f}j})^2}{z_{f\bar{f}j}(1 - z_{f\bar{f}j})} < y_{f\bar{f}j} < \frac{(k^0)^2}{\bar{P}^2} z_{f\bar{f}j}(1 - z_{f\bar{f}j}) \Delta\theta^2, \quad 0 < z_{f\bar{f}j} < 1. \tag{4.26}$$

## 5 Collinear singularities from $f \rightarrow f\gamma^*$ splittings

### 5.1 Asymptotics in the collinear limit

We consider a generic scattering process

$$f(p_f, \kappa_f) + a(p_a) \rightarrow f(p'_f) + X, \tag{5.1}$$

with the momenta of the particles and the (sign of the) helicity  $\kappa_f = \pm$  of the incoming fermion  $f$  indicated in parentheses. We are interested in the region where the squared momentum transfer  $(p_f - p'_f)^2 = 2m_f^2 - 2p_f p'_f$  of the scattered fermion  $f$  becomes of the order  $\mathcal{O}(m_f^2)$ , where  $m_f$  is small compared to typical scales in the process. The singular behaviour of the full squared matrix element  $|\mathcal{M}_{fa \rightarrow fX}(p_f, p'_f; \kappa_f)|^2$  entirely originates from diagrams containing an  $f \rightarrow f\gamma^*$  splitting, i.e. the singularity is related to the subprocess  $\gamma a \rightarrow X$ . For the matrix element of this subprocess we write  $\mathcal{M}_{\gamma a \rightarrow X}(\tilde{k}, p_a, \lambda_\gamma) = T_{\gamma a \rightarrow X}^\mu(\tilde{k}) \varepsilon_{\lambda_\gamma, \mu}(\tilde{k})$ , where  $T_{\gamma a \rightarrow X}^\mu(\tilde{k})$  is the amplitude without the photon polarization vector  $\varepsilon_{\lambda_\gamma, \mu}(\tilde{k})$ . In the collinear limit  $p_f p'_f \rightarrow 0$  the momentum  $\tilde{k}$  is given by  $k = p_f - p'_f$  up to mass-suppressed terms. Neglecting terms that are irrelevant in the limit  $m_f \rightarrow 0$  the squared matrix element  $|\mathcal{M}_{fa \rightarrow fX}(p_f, p'_f; \kappa_f)|^2$  asymptotically behaves like

$$|\mathcal{M}_{fa \rightarrow fX}(p_f, p_a, p'_f; \kappa_f)|^2 \xrightarrow{p_f p'_f \rightarrow 0} N_{c,f} Q_f^2 e^2 h_{\kappa_f, \mu\nu}^{ff}(p_f, p'_f) T_{\gamma a \rightarrow X}^\mu(\tilde{k}, p_a)^* T_{\gamma a \rightarrow X}^\nu(\tilde{k}, p_a), \tag{5.2}$$

where

$$\begin{aligned}
h_{\kappa_f, \mu\nu}^{ff}(p_f, p'_f) &= \frac{-1}{(p_f - p'_f)^2} \left[ -g_{\mu\nu} - \frac{4(1-x)}{x^2} \frac{k_{\perp, \mu} k_{\perp, \nu}}{k_{\perp}^2 - x^2 m_f^2} \right. \\
&\quad \left. + \frac{\kappa_f}{x} \left( 2 - x + \frac{2x^2 m_f^2}{(p_f - p'_f)^2} \right) (\varepsilon_{+, \mu}(\tilde{k})^* \varepsilon_{+, \nu}(\tilde{k}) - \varepsilon_{-, \mu}(\tilde{k})^* \varepsilon_{-, \nu}(\tilde{k})) \right]
\end{aligned} \tag{5.3}$$

with

$$x = \frac{k^0}{p_f^0}. \tag{5.4}$$

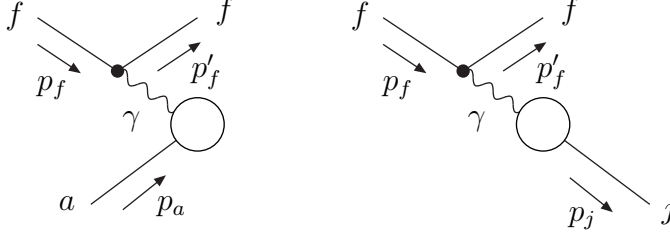


Figure 4: Generic diagrams for the splittings  $f \rightarrow f\gamma^*$  with an initial-state spectator  $a$  or a final-state spectator  $j$ , where  $f$  is a light fermion or antifermion.

The momentum  $k_\perp$  is the component of  $k$  that is orthogonal to the collinear axis defined by  $p_f$ , i.e.  $k_\perp p_f = 0$ , and becomes of  $\mathcal{O}(m_f)$  in the collinear limit. A derivation of this factorization is described in App. D.1. Note that  $h_{\kappa_f, \mu\nu}^{ff}$  in Eq. (5.2) is not proportional to the polarization sum  $E_{\mu\nu} = \sum_{\lambda_\gamma} \varepsilon_{\lambda_\gamma, \mu}(\tilde{k})^* \varepsilon_{\lambda_\gamma, \nu}(\tilde{k})$  of the photon, so that the r.h.s. is not proportional to the polarization-summed squared amplitude  $|\mathcal{M}_{\gamma a \rightarrow X}|^2$  of the subprocess. This spin correlation has to be taken into account in the construction of an appropriate subtraction function in order to guarantee a point-wise cancellation of the singular behaviour in the collinear phase-space region. The spin correlation encoded in  $h_{\kappa_f, \mu\nu}^{ff}$  drops out if the average over the azimuthal angle  $\phi'_f$  of the  $f \rightarrow f\gamma$  splitting plane around the collinear axis is taken. Details of this averaging process, which is indicated by  $\langle \dots \rangle_{\phi'_f}$ , are given in App. D.1. The result is

$$\langle |\mathcal{M}_{fa \rightarrow fX}(p_f, p_a, p'_f; \kappa_f)|^2 \rangle_{\phi'_f} \widetilde{p_f p'_f \rightarrow 0} N_{c,f} Q_f^2 e^2 h_\tau^{ff}(p_f, p'_f) |\mathcal{M}_{\gamma a \rightarrow X}(\tilde{k}, p_a; \lambda_\gamma = \tau \kappa_f)|^2 \quad (5.5)$$

with summation over  $\tau = \pm$  and

$$h_\tau^{ff}(p_f, p'_f) = \frac{-1}{x(p_f - p'_f)^2} \left[ P_{\gamma f}(x) + \frac{2xm_f^2}{(p_f - p'_f)^2} + \tau \left( 2 - x + \frac{2x^2 m_f^2}{(p_f - p'_f)^2} \right) \right], \quad (5.6)$$

which is valid in four space-time dimensions up to terms that are further suppressed by factors of  $m_f$ . Here  $P_{\gamma f}(x)$  is the splitting function

$$P_{\gamma f}(x) = \frac{1 + (1 - x)^2}{x}. \quad (5.7)$$

Since the collinear singularity for  $p_f p'_f \rightarrow 0$  can be attributed to a single leg (the photon) of the related hard process  $\gamma a \rightarrow X$ , also in this case there is no need to construct the subtraction function  $|\mathcal{M}_{\text{sub}}|^2$  from several dipole contributions. The function  $|\mathcal{M}_{\text{sub}}|^2$  can be chosen as a single term  $\propto Q_f^2$ , and a spectator is only used in the phase-space construction as previously. In the following we again describe the “dipole” construction in two variants: one with a spectator from the initial state, another with a spectator from the final state. The two situations are illustrated in Fig. 4.

## 5.2 Initial-state spectator

We define the subtraction function as

$$|\mathcal{M}_{\text{sub}}(\kappa_f)|^2 = N_{c,f} Q_f^2 e^2 h_{\kappa_f, \mu\nu}^{ff,a}(p_f, p'_f, p_a) T_{\gamma a \rightarrow X}^\mu(\tilde{k}, p_a)^* T_{\gamma a \rightarrow X}^\nu(\tilde{k}, p_a) \quad (5.8)$$



with

$$h_{\kappa_f}^{ff,a,\mu\nu}(p_f, p'_f, p_a) = \frac{-1}{(p_f - p'_f)^2} \left[ -g^{\mu\nu} - \frac{4(1 - x_{f,fa})}{x_{f,fa}^2} \frac{\tilde{k}_\perp^\mu \tilde{k}_\perp^\nu}{\tilde{k}_\perp^2 - x_{f,fa}^2 m_f^2} \right. \\ \left. + \frac{\kappa_f}{x} \left( 2 - x + \frac{2x^2 m_f^2}{(p_f - p'_f)^2} \right) (\varepsilon_+^\mu(\tilde{k})^* \varepsilon_+^\nu(\tilde{k}) - \varepsilon_-^\mu(\tilde{k})^* \varepsilon_-^\nu(\tilde{k})) \right] \quad (5.9)$$

and the auxiliary parameters

$$x_{f,fa} = \frac{p_a p_f - p_f p'_f - p_a p'_f + m_f^2}{p_a p_f}, \quad y_{f,fa} = \frac{p_f p'_f - m_f^2}{p_a p_f}. \quad (5.10)$$

Assuming again the incoming particle  $a$  to be massless and defining

$$s = (p_f + p_a)^2 = m_f^2 + 2p_a p_f = \bar{s} + m_f^2, \quad P^\mu = p_f^\mu + p_a^\mu - p'_f{}^\mu, \quad (5.11)$$

the needed auxiliary momenta for the related process  $\gamma a \rightarrow X$  are given by

$$\tilde{k}^\mu(x) = x \left( p_f^\mu - \frac{m_f^2}{\bar{s}} p_a^\mu \right), \quad \tilde{k}^\mu = \tilde{k}^\mu(x_{f,fa}), \quad \tilde{k}_\perp^\mu = p_f'^\mu - \frac{p_f' \tilde{k}}{p_a \tilde{k}} p_a^\mu, \\ \tilde{P}^\mu(x) = \tilde{k}^\mu(x) + p_a^\mu, \quad \tilde{P}^\mu = \tilde{P}^\mu(x_{f,fa}), \\ \tilde{k}_n^\mu = \Lambda^\mu{}_\nu k_n^\nu, \quad (5.12)$$

where the Lorentz transformation matrix  $\Lambda^\mu{}_\nu$  is constructed from the momenta  $P^\mu$  and  $\tilde{P}^\mu$  as in Eq. (3.10). In these equations we kept the dependence on  $m_f$ , but of course in the numerical integration of  $|\mathcal{M}_{fa \rightarrow fX}|^2 - |\mathcal{M}_{\text{sub}}|^2$  we can set  $m_f$  to zero if we are only interested in the limit  $m_f \rightarrow 0$ . For the integration of  $|\mathcal{M}_{\text{sub}}|^2$  over the collinear-singular region, we need the  $m_f$ -dependence of its azimuthal average,

$$\langle |\mathcal{M}_{\text{sub}}(\kappa_f)|^2 \rangle_{\phi'_f} = N_{c,f} Q_f^2 e^2 h_\tau^{ff,a}(p_f, p'_f, p_a) |\mathcal{M}_{\gamma a \rightarrow X}(\tilde{k}, p_a; \lambda_\gamma = \tau \kappa_f)|^2 \quad (5.13)$$

with summation over  $\tau = \pm$  and

$$h_\tau^{ff,a}(p_f, p'_f, p_a) = \frac{1}{\bar{s}xy} \left[ P_{\gamma f}(x) - \frac{2x(1-x)m_f^2}{y[\bar{s}(1-x-y) - m_f^2(2x+y)]} + \tau \left( 2 - x - \frac{2x^2 m_f^2}{\bar{s}y} \right) \right], \quad (5.14)$$

where we have used the shorthands  $x = x_{f,fa}$  and  $y = y_{f,fa}$ . An appropriate phase-space splitting is given by

$$\int d\phi(p'_f, P; p_f + p_a) = \int_0^{x_1} dx \int d\phi(\tilde{P}(x); \tilde{k}(x) + p_a) \int [dp'_f(s, x, y)] \quad (5.15)$$

with the explicit form of  $\int [dp'_f]$

$$\int [dp'_f(s, x, y)] = \frac{\bar{s}}{4(2\pi)^3} \int_{y_1(x)}^{y_2(x)} dy \int d\phi'_f \quad (5.16)$$

and the integration limits for the variables  $x$  and  $y$

$$x_1 = \frac{\sqrt{\bar{s}} - m_f}{\sqrt{\bar{s}} + m_f}, \quad y_{1,2}(x) = \frac{\bar{s}}{2s} \left( 1 - x - \frac{2m_f^2}{\bar{s}} x \mp \sqrt{(1-x)^2 - \frac{4m_f^2}{\bar{s}} x} \right). \quad (5.17)$$

In the limit  $m_f \rightarrow 0$  the integral

$$\mathcal{H}_\tau^{ff,a}(s, x) = \frac{x\bar{s}}{2} \int_{y_1(x)}^{y_2(x)} dy h_\tau^{ff,a}(p_f, p'_f, p_a) \quad (5.18)$$

can be easily evaluated to

$$\mathcal{H}_\tau^{ff,a}(s, x) = \ln\left(\frac{\sqrt{s}(1-x)}{xm_f}\right) [P_{\gamma f}(x) + \tau(2-x)] - \frac{1-x}{x} - \tau(1-x), \quad (5.19)$$

and the part to be added to the cross section reads

$$\sigma_{fa \rightarrow fX}^{\text{sub}}(p_f, p_a; \kappa_f) = \frac{Q_f^2 \alpha}{2\pi} \int_0^1 dx \mathcal{H}_\tau^{ff,a}(s, x) \sigma_{\gamma a \rightarrow X}(\tilde{k} = xp_f, p_a; \lambda_\gamma = \tau \kappa_f). \quad (5.20)$$

### 5.3 Final-state spectator

As an alternative to the case of an initial-state spectator, we now present the treatment with a final-state spectator  $j$ , i.e. we consider the process

$$f(p_f, \kappa_f) + a(p_a) \rightarrow f(p'_f) + j(p_j) + X. \quad (5.21)$$

The particles  $a$  and  $j$  are assumed massless in the following; the case of a massive spectator  $j$  is described in App. D.2. The subtraction function is constructed as follows,

$$|\mathcal{M}_{\text{sub}}(\kappa_f)|^2 = N_{c,f} Q_f^2 e^2 h_{j,\kappa_f,\mu\nu}^{ff}(p_f, p'_f, p_j) T_{\gamma a \rightarrow jX}^\mu(\tilde{k}, p_a, \tilde{p}_j)^* T_{\gamma a \rightarrow jX}^\nu(\tilde{k}, p_a, \tilde{p}_j) \quad (5.22)$$

with

$$h_{j,\kappa_f}^{ff,\mu\nu}(p_f, p'_f, p_j) = \frac{-1}{(p_f - p'_f)^2} \left[ -g^{\mu\nu} - \frac{4(\bar{z}_{fj,f} p_f^\mu - z_{fj,f} p_j^\mu)(\bar{z}_{fj,f} p_f^\nu - z_{fj,f} p_j^\nu)}{(p_f - p'_f)^2 x_{fj,f}^2 \bar{z}_{fj,f}} \right. \\ \left. + \frac{\kappa_f}{x} \left( 2 - x + \frac{2x^2 m_f^2}{(p_f - p'_f)^2} \right) (\varepsilon_+^\mu(\tilde{k})^* \varepsilon_+^\nu(\tilde{k}) - \varepsilon_-^\mu(\tilde{k})^* \varepsilon_-^\nu(\tilde{k})) \right] \quad (5.23)$$

and the auxiliary parameters

$$x_{fj,f} = \frac{p_f p'_f + p_f p_j - p'_f p_j}{p_f p'_f + p_f p_j}, \quad z_{fj,f} = 1 - \bar{z}_{fj,f} = \frac{p_f p'_f}{p_f p'_f + p_f p_j}. \quad (5.24)$$

The momenta  $\tilde{k}$  and  $\tilde{p}_j$  are given by

$$\tilde{k}^\mu = x_{fj,f} p_f^\mu, \quad \tilde{p}_j = P^\mu + \tilde{k}^\mu, \quad P^\mu = p_f'^\mu + p_j^\mu - p_f^\mu. \quad (5.25)$$

Note that this construction of momenta is based on the restriction  $m_f = 0$ , which is used in the integration of the difference  $|\mathcal{M}_{fa \rightarrow fjX}|^2 - |\mathcal{M}_{\text{sub}}|^2$  for  $m_f \rightarrow 0$ .

In the integration of  $|\mathcal{M}_{\text{sub}}|^2$  over the collinear-singular phase space the correct dependence on a finite  $m_f$  is required. We sketch this procedure in App. D.2 for a possibly finite spectator mass  $m_j$ , but here we give only the relevant formulas needed in applications. The cross-section contribution  $\sigma_{fa \rightarrow fjX}^{\text{sub}}$  that has to be added to the integrated difference  $|\mathcal{M}_{fa \rightarrow fjX}|^2 - |\mathcal{M}_{\text{sub}}|^2$  is given by

$$\sigma_{fa \rightarrow fjX}^{\text{sub}}(p_f, p_a; \kappa_f) = \frac{Q_f^2 \alpha}{2\pi} \int_0^1 dx \mathcal{H}_{ffj,\tau}(P^2, x) \sigma_{\gamma a \rightarrow jX}(\tilde{k} = xp_f, p_a; \lambda_\gamma = \tau \kappa_f), \quad (5.26)$$

where the collinear singularity is contained in the kernels

$$\mathcal{H}_{ffj,\tau}(P^2, x) = \frac{1}{2} \ln\left(\frac{-P^2(1-x)}{x^3 m_f^2}\right) [P_{\gamma f}(x) + \tau(2-x)] - \frac{1-x}{x} - \tau(1-x). \quad (5.27)$$

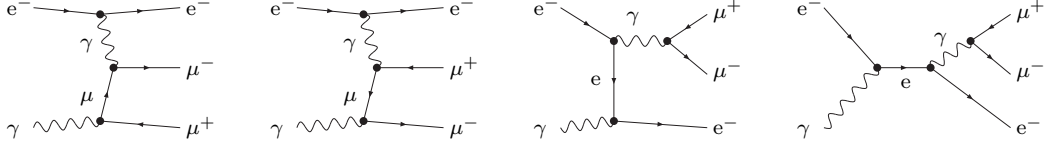


Figure 5: QED diagrams contributing to  $e^-\gamma \rightarrow e^-\mu^-\mu^+$  at tree level.

#### 5.4 Phase-space slicing

Finally, we derive the integral over the collinear phase-space region for the slicing approach. This region is defined by restricting the angle  $\theta'_f$  between the outgoing and incoming  $f$  to small values,  $\theta'_f < \Delta\theta \ll 1$ .

In Section 5.2 this restriction leads to new limits in  $y_{f,fa}$ ,

$$\frac{m_f^2}{s} \frac{x_{f,fa}^2}{1 - x_{f,fa}} < y_{f,fa} < \frac{(p_f^0)^2}{s} (1 - x_{f,fa}) \Delta\theta^2, \quad (5.28)$$

which modify the integrated result to

$$\mathcal{H}_\tau^{ff}(p_f^0, x) = \ln\left(\frac{p_f^0(1-x)\Delta\theta}{xm_f}\right) [P_{\gamma f}(x) + \tau(2-x)] - \frac{1-x}{x} - \tau(1-x), \quad (5.29)$$

where the integral is defined analogously to Eq. (5.18). The cross-section contribution for the collinear scattering of  $f$  is given by

$$\sigma_{fa \rightarrow fX}^{\text{coll},f}(p_f, p_a; \kappa_f) = \frac{Q_f^2 \alpha}{2\pi} \int_0^1 dx \mathcal{H}_\tau^{ff}(p_f^0, x) \sigma_{\gamma a \rightarrow X}(\tilde{k} = xp_f, p_a; \lambda_\gamma = \tau \kappa_f). \quad (5.30)$$

The same results can be obtained from Section 5.3 with App. D.2, where the new limits on the integration variables are given by

$$\frac{m_f^2}{-\bar{P}^2} \frac{x_{fj,f}[1 + (1 - x_{fj,f})^2]}{1 - x_{fj,f}} < z_{fj,f} < \frac{(p_f^0)^2}{-\bar{P}^2} x_{fj,f}(1 - x_{fj,f}) \Delta\theta^2. \quad (5.31)$$

### 6 Application to the process $e^-\gamma \rightarrow e^-\mu^-\mu^+$

In this section we illustrate the application of the methods described in Sections 3, 4, and 5 to the process  $e^-\gamma \rightarrow e^-\mu^-\mu^+$  at a centre-of-mass energy  $\sqrt{s}$  much larger than the involved particle masses,  $\sqrt{s} \gg m_e, m_\mu$ . Of course, this process is not of particular importance in particle phenomenology, but it involves the three issues of (i) incoming photons splitting into light  $f\bar{f}$  pairs, (ii) the collinear production of light  $f\bar{f}$  pairs, and (iii) forward-scattered fermions and, thus, provides a good test process for these cases. As already mentioned in Section 2, our treatment of non-collinear-safe final-state radiation has already been tested in other processes.

To illustrate the formalism, it is sufficient to consider the process  $e^-\gamma \rightarrow e^-\mu^-\mu^+$  in QED, where only the four diagrams shown in Fig. 5 contribute. The corresponding helicity amplitudes, including the full dependence on the masses  $m_e$  and  $m_\mu$ , can be

obtained from the treatment of  $e^- \gamma \rightarrow e^- e^- e^+$  presented in Ref. [22] after some obvious substitutions. In the following we compare the result with the full mass dependence to results obtained with the described subtraction and slicing methods in various kinematical situations. Denoting the polar angle of an outgoing particle  $i$  by  $\theta_i$  and the angle between the two outgoing muons by  $\alpha_{\mu\mu}$ , we distinguish the following cases:

a) *No collinear splittings*

Angular cuts:  $\theta_{\text{cut}} < \theta_{e^-} < 180^\circ - \theta_{\text{cut}}$  and  $\theta_{\mu^\pm} < 180^\circ - \theta_{\text{cut}}$  and  $\theta_{\text{cut}} < \alpha_{\mu\mu}$ .

No collinear singularities are included, and the integrated cross section is well defined for vanishing fermion masses, i.e. none of the subtraction methods has to be applied. The difference between massive and massless calculations indicates the size of the fermion mass effects.

b) *Collinear splitting  $\gamma \rightarrow e^- e^{+*}$*

Angular cuts:  $\theta_{\text{cut}} < \theta_{e^-}$  and  $\theta_{\mu^\pm} < 180^\circ - \theta_{\text{cut}}$  and  $\theta_{\text{cut}} < \alpha_{\mu\mu}$ .

The collinear splitting  $\gamma \rightarrow e^- e^{+*}$  of the incoming photon is integrated over, so that the third diagram of Fig. 5 develops a collinear singularity for backward-scattered electrons. The methods of Section 3 are applied to the calculation with massless fermions.

c) *Collinear splittings  $\gamma \rightarrow \mu^\mp \mu^{\pm*}$*

Angular cuts:  $\theta_{\text{cut}} < \theta_{e^-} < 180^\circ - \theta_{\text{cut}}$  and  $\theta_{\text{cut}} < \alpha_{\mu\mu}$ .

The collinear splittings  $\gamma \rightarrow \mu^\mp \mu^{\pm*}$  of the incoming photon are integrated over, so that the first two diagrams of Fig. 5 develop collinear singularities for backward-scattered muons. The methods of Section 3 are applied to the calculation with massless fermions.

d) *Collinear splitting  $\gamma^* \rightarrow \mu^- \mu^+$*

Angular cuts:  $\theta_{\text{cut}} < \theta_{e^-} < 180^\circ - \theta_{\text{cut}}$  and  $\theta_{\mu^\pm} < 180^\circ - \theta_{\text{cut}}$ .

The collinear splitting  $\gamma^* \rightarrow \mu^- \mu^+$  of an intermediate photon is integrated over, so that the last two diagrams of Fig. 5 develop collinear singularities for collinearly produced muons. The methods of Section 4 are applied to the calculation with massless fermions.

e) *Collinear splitting  $e^- \rightarrow e^- \gamma^*$*

Angular cuts:  $\theta_{e^-} < 180^\circ - \theta_{\text{cut}}$  and  $\theta_{\mu^\pm} < 180^\circ - \theta_{\text{cut}}$  and  $\theta_{\text{cut}} < \alpha_{\mu\mu}$ .

The collinear splitting  $e^- \rightarrow e^- \gamma^*$  of the incoming electron is integrated over, so that the first two diagrams of Fig. 5 develop collinear singularities for forward-scattered electrons. The methods of Section 5 are applied to the calculation with massless fermions.

For the numerical evaluation we set the fermion masses to  $m_e = 0.51099907 \text{ MeV}$  and  $m_\mu = 0.10565839 \text{ GeV}$ , the fine-structure constant to  $\alpha = e^2/(4\pi) = 1/137.0359895$ , the beam energies to  $E = E_e = E_\gamma = 250 \text{ GeV}$ , and the angular cut to  $\theta_{\text{cut}} = 10^\circ$ . In the subtraction and slicing methods the masses  $m_e$  and  $m_\mu$  are neglected everywhere except for the mass-singular logarithms, i.e. the laboratory frame defined by the above beam energies coincides with the centre-of-mass system. For the fully massive calculation the

two frames are connected by a (numerically irrelevant) boost along the beam axis with a tiny boost velocity of  $\mathcal{O}(m_e^2/E^2)$ . Our numerical results for the different kinematical situations and the various methods are collected in Table 1. In addition in Table 2 we show the analogous results for the situation where the energy of each final-state lepton  $l = e^-, \mu^\pm$  is restricted by  $E_l > 10 \text{ GeV}$ . All results are obtained with an integration by Vegas [24], using  $25 \times 10^6$  events. While a simple phase-space parametrization is sufficient in the subtraction formalism, dedicated phase-space mappings are required to flatten the corresponding collinear poles in the slicing approach and when employing the full mass dependence of the matrix elements. The fully massive results have been checked with the program WHIZARD [25], where agreement within the integration errors has been found.

The results obtained with the different subtraction variants, where a spectator is chosen from the initial state (IS) or from the final state (FS), are in mutual agreement within the integration error, which is indicated in parentheses. Subtraction and slicing results are also consistent within the statistical errors as long as the angular slicing cut  $\Delta\theta$  is not chosen too large. For example, some of the slicing results for  $\Delta\theta = 10^{-1}$  still show a significant residual dependence on  $\Delta\theta$ . In the chosen example, the integration errors of the subtraction and slicing results are of the same order of magnitude. However, we would like to mention that the subtraction approach is often more efficient, as e.g. observed in the applications of Refs. [14–17,21] mentioned above. This superiority of the subtraction formalism typically deteriorates if complicated phase-space cuts are applied, as in the chosen example, because the cuts act differently in the various auxiliary phase spaces and thus introduce new peak structures in the integrand.

Finally, we remark that the impact of mass-suppressed terms is significantly reduced if the cut on the lepton energies  $E_l$  is applied. This cut guarantees that  $E_l \gg m_l$  overall in phase space, so that mass-suppressed terms are proportional to  $m_l^2/Q^2$  with  $Q \gg m_l$ . Without any restriction on  $E_l$ , there are at least small regions of phase space where  $Q$  is not much smaller than  $m_l$ , leading to larger mass effects. This feature is clearly visible in Tables 1 and 2 when comparing results based on the full mass dependence in the matrix elements with the subtraction and slicing results that are based on the asymptotic limit  $m_l \rightarrow 0$ .

## 7 Summary

The dipole subtraction formalism for photonic corrections is extended to various photon–fermion splittings where the resulting collinear singularities lead to corrections that are enhanced by logarithms of small fermion masses  $m_f$ . Specifically, we have considered non-collinear-safe final-state radiation, collinear fermion production from incoming photons, forward-scattered incoming fermions, and collinearly produced fermion–antifermion pairs. All formulas needed in applications are provided, only the scattering matrix elements for the underlying process and for relevant subprocesses have to be supplemented in the simple approximation of a massless fermion  $f$ . Particle polarization is taken care of in all relevant cases, e.g., for incoming fermions and photons. For the purpose of cross-checking results in applications, we also provide the formulas needed in the phase-space slicing method.

Collinear splittings	Method	$\sigma_{+-}[\text{pb}]$	$\sigma_{++}[\text{pb}]$
a) none	full mass dependence	0.50910(9)	0.47172(6)
	massless case	0.51110(9)	0.47384(7)
b) $\gamma \rightarrow e^- e^{+*}$	full mass dependence	0.52213(7)	0.56762(7)
	subtraction (IS spectator)	0.52424(8)	0.57027(8)
	subtraction (FS spectator)	0.52434(7)	0.57017(9)
	slicing ( $\Delta\theta = 10^{-1}$ )	0.52410(7)	0.57021(6)
	slicing ( $\Delta\theta = 10^{-3}$ )	0.52431(9)	0.57021(7)
	slicing ( $\Delta\theta = 10^{-5}$ )	0.52423(8)	0.57028(7)
c) $\gamma \rightarrow \mu^\mp \mu^{\pm*}$	full mass dependence	2.5890(5)	2.3615(4)
	subtraction (IS spectator)	2.5872(3)	2.3586(5)
	subtraction (FS spectator)	2.5873(8)	2.3585(5)
	slicing ( $\Delta\theta = 10^{-1}$ )	2.5883(3)	2.3609(2)
	slicing ( $\Delta\theta = 10^{-3}$ )	2.5859(8)	2.3578(8)
	slicing ( $\Delta\theta = 10^{-5}$ )	2.5876(13)	2.3572(13)
d) $\gamma^* \rightarrow \mu^- \mu^+$	full mass dependence	0.54076(8)	0.53357(8)
	subtraction (IS spectator)	0.54309(8)	0.53597(7)
	subtraction (FS spectator)	0.54306(8)	0.53603(7)
	slicing ( $\Delta\theta = 10^{-1}$ )	0.53164(19)	0.52386(16)
	slicing ( $\Delta\theta = 10^{-3}$ )	0.54287(17)	0.53624(15)
	slicing ( $\Delta\theta = 10^{-5}$ )	0.54335(18)	0.53580(18)
e) $e^- \rightarrow e^- \gamma^*$	full mass dependence	5.5465(7)	4.7060(6)
	subtraction (IS spectator)	5.5495(4)	4.7070(3)
	subtraction (FS spectator)	5.5484(6)	4.7064(5)
	slicing ( $\Delta\theta = 10^{-1}$ )	5.5313(1)	4.6880(1)
	slicing ( $\Delta\theta = 10^{-3}$ )	5.5488(3)	4.7071(3)
	slicing ( $\Delta\theta = 10^{-5}$ )	5.5486(5)	4.7067(4)

Table 1: QED cross sections  $\sigma_{\kappa\lambda}$  for  $e^- \gamma \rightarrow e^- \mu^- \mu^+$  in the various setups described in the main text, with signs  $\kappa$  and  $\lambda$  of the helicities of the incoming  $e^-$  and  $\gamma$ , respectively. The results are obtained with the indicated methods, where IS and FS stand for spectators in the initial and final states, respectively.

Collinear splittings	Method	$\sigma_{+-}[\text{pb}]$	$\sigma_{++}[\text{pb}]$
a) none	full mass dependence	0.45780(6)	0.41699(6)
	massless case	0.45779(6)	0.41704(5)
b) $\gamma \rightarrow e^-e^{+*}$	full mass dependence	0.46995(6)	0.50351(6)
	subtraction (IS spectator)	0.46999(6)	0.50345(6)
	subtraction (FS spectator)	0.46995(6)	0.50348(7)
	slicing ( $\Delta\theta = 10^{-1}$ )	0.46990(7)	0.50349(5)
	slicing ( $\Delta\theta = 10^{-3}$ )	0.46992(7)	0.50352(5)
	slicing ( $\Delta\theta = 10^{-5}$ )	0.46992(7)	0.50355(6)
c) $\gamma \rightarrow \mu^\mp \mu^{\pm*}$	full mass dependence	2.4934(5)	2.2637(4)
	subtraction (IS spectator)	2.4931(3)	2.2637(2)
	subtraction (FS spectator)	2.4923(6)	2.2642(5)
	slicing ( $\Delta\theta = 10^{-1}$ )	2.4895(2)	2.2606(2)
	slicing ( $\Delta\theta = 10^{-3}$ )	2.4917(7)	2.2628(7)
	slicing ( $\Delta\theta = 10^{-5}$ )	2.4905(12)	2.2626(12)
d) $\gamma^* \rightarrow \mu^- \mu^+$	full mass dependence	0.48606(7)	0.47396(8)
	subtraction (IS spectator)	0.48620(7)	0.47407(6)
	subtraction (FS spectator)	0.48630(6)	0.47401(6)
	slicing ( $\Delta\theta = 10^{-1}$ )	0.47588(19)	0.46363(13)
	slicing ( $\Delta\theta = 10^{-3}$ )	0.48607(19)	0.47399(14)
	slicing ( $\Delta\theta = 10^{-5}$ )	0.48623(20)	0.47425(15)
e) $e^- \rightarrow e^- \gamma^*$	full mass dependence	5.4878(6)	4.6467(5)
	subtraction (IS spectator)	5.4866(3)	4.6471(3)
	subtraction (FS spectator)	5.4871(5)	4.6475(5)
	slicing ( $\Delta\theta = 10^{-1}$ )	5.4690(1)	4.6278(1)
	slicing ( $\Delta\theta = 10^{-3}$ )	5.4869(3)	4.6467(3)
	slicing ( $\Delta\theta = 10^{-5}$ )	5.4862(5)	4.6466(4)

Table 2: Same as in Table 1, but with an energy cut  $E_l > 10 \text{ GeV}$  for all final-state leptons.

As an example illustrating the use and performance of the proposed methods we have explicitly applied the subtraction procedures to the process  $e^- \gamma \rightarrow e^- \mu^- \mu^+$  and compared the results to those obtained with phase-space slicing. The presented subtraction variants will certainly be used in several precision calculations needed for present and future collider experiments such as the LHC or ILC.

## Acknowledgement

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## Appendix

### A More details on non-collinear-safe final-state radiation

Here we generalize the results of Section 2.1, where non-collinear-safe photon radiation off fermions is treated, to the situation where massive spectators in the final state exist. To this end, we only have to consider the case of final-state emitter and final-state spectator.

For  $m_i \rightarrow 0$ ,  $m_\gamma = 0$ , but  $m_j \neq 0$ , the boundary of the  $y_{ij}$  integration [given for the massless case in Eq. (2.8)] is given by

$$y_1(z) = \frac{m_i^2(1-z)}{\bar{P}_{ij}^2 z},$$

$$y_2(z) = \left[ \xi(z) + 1 + \sqrt{\xi(z)[\xi(z) + 2]} \right]^{-1} \quad \text{with} \quad \xi(z) = \frac{m_j^2}{2\bar{P}_{ij}^2 z(1-z)}, \quad (\text{A.1})$$

and the functions relevant for the integrand  $g_{ij,\tau}^{(\text{sub})}$  behave as

$$p_{ik} = \frac{\bar{P}_{ij}^2 y_{ij}}{2}, \quad \bar{P}_{ij}^2 = P_{ij}^2 - m_j^2,$$

$$R_{ij}(y) = \sqrt{(1-y)^2 - \frac{4m_j^2 y}{\bar{P}_{ij}^2}}, \quad r_{ij}(y) = 1 - \frac{2m_j^2}{\bar{P}_{ij}^2} \frac{y}{1-y}. \quad (\text{A.2})$$

The evaluation of Eq. (2.6) now becomes non-trivial and yields

$$\begin{aligned} \bar{\mathcal{G}}_{ij,+}^{(\text{sub})}(P_{ij}^2, z) &= -P_{ff}(z) \ln\left(\frac{m_i^2}{z\bar{P}_{ij}^2}[1 - \eta(z)]\right) + (1+z) \ln(1-z) - \frac{2z}{1-z} \\ &\quad + (1+z) \ln\left(1 + \frac{m_j^2}{\bar{P}_{ij}^2 \eta(z)}\right) - \frac{2}{(1-z)\sigma(z)} \left\{ \ln\left(1 + \frac{\bar{P}_{ij}^2 \eta(z)[1 - z\eta(z)]}{m_j^2(1-z)}\right) \right. \\ &\quad \left. - 2 \ln\left(1 - \frac{2z\eta(z)}{1 + \sigma(z)}\right) + \sigma(z) \ln\left(\frac{m_j^2}{\bar{P}_{ij}^2 \eta(z)}(1-z)\right) \right\} - \bar{\mathcal{G}}_{ij,-}^{(\text{sub})}(P_{ij}^2, z), \\ \bar{\mathcal{G}}_{ij,-}^{(\text{sub})}(P_{ij}^2, z) &= 1 - z, \end{aligned} \quad (\text{A.3})$$



with the auxiliary functions

$$\sigma(z) = \sqrt{1 + \frac{4m_j^2}{\bar{P}_{ij}^2} z(1-z)}, \quad \eta(z) = \begin{cases} [1 - y_2(z)]z & \text{for } z < \frac{1}{2}, \\ [1 - y_2(z)](1-z) & \text{for } z > \frac{1}{2}. \end{cases} \quad (\text{A.4})$$

For  $m_j \rightarrow 0$ , the results for  $\bar{\mathcal{G}}_{ij,\tau}^{(\text{sub})}(P_{ij}^2, z_{ij})$  reduce to Eq. (2.10), as can be easily seen after realizing that  $\eta(z) = \mathcal{O}(m_j)$  and  $\sigma(z) = 1 + \mathcal{O}(m_j^2)$  in this limit.

## B More details on the subtraction for $\gamma \rightarrow f\bar{f}^*$ splittings

### B.1 Factorization in the collinear limit

In this section we derive the asymptotic behaviour (3.3) of the squared amplitude  $|\mathcal{M}_{\gamma a \rightarrow fX}|^2$  for the case where the outgoing light fermion flies along the direction of the incoming photon. We consider polarized incoming photons with momentum  $k^\mu$  and polarization vector  $\varepsilon_{\lambda_\gamma}^\mu$ , where  $\lambda_\gamma = \pm$  is the sign of its helicity. We further introduce a light-like gauge vector  $n^\mu$  ( $n^2 = 0, nk \neq 0$ ), i.e.  $\varepsilon_{\lambda_\gamma}^\mu$  is characterized by

$$\varepsilon_{-\lambda_\gamma}^\mu = (\varepsilon_{\lambda_\gamma}^\mu)^*, \quad k\varepsilon_{\lambda_\gamma} = n\varepsilon_{\lambda_\gamma} = 0. \quad (\text{B.1})$$

In the following we make use of the identity<sup>6</sup>

$$\varepsilon_\pm^\mu (\varepsilon_\pm^\nu)^* = \varepsilon_\pm^\mu \varepsilon_\mp^\nu = \frac{1}{2} E^{\mu\nu}(k) \mp \frac{i}{2kn} \epsilon^{\mu\nu\rho\sigma} k_\rho n_\sigma, \quad (\text{B.2})$$

where

$$E^{\mu\nu}(k) = \varepsilon_+^\mu \varepsilon_-^\nu + \varepsilon_-^\mu \varepsilon_+^\nu = -g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{kn} \quad (\text{B.3})$$

is the polarization sum of the photon in four space-time dimensions and  $\epsilon^{\mu\nu\rho\sigma}$  the Levi-Civita tensor with  $\epsilon^{0123} = +1$ .

In a gauge for the photon where  $nk = \mathcal{O}(k^0)$ , it is easily shown by power counting that the logarithmic singularity arising from the phase-space region  $kp_f = \mathcal{O}(m_f^2)$  ( $m_f \ll k^0$ ) originates from diagrams in which the incoming photon collinearly splits into a light  $f\bar{f}^*$  pair. The generic form of such graphs is shown in Fig. 6. Assuming summation over the polarization of the outgoing fermion  $f$ , the squared matrix element, thus, behaves like

$$|\mathcal{M}_{\gamma a \rightarrow fX}(k, p_a, p_f; \lambda_\gamma)|^2 \underset{kp_f \rightarrow 0}{\sim} Q_f^2 e^2 \bar{T}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a) \frac{-\not{p}_{\bar{f}} + m_{\bar{f}}}{p_{\bar{f}}^2 - m_{\bar{f}}^2} \not{\varepsilon}_{\lambda_\gamma}^* (\not{p}_f + m_f) \not{\varepsilon}_{\lambda_\gamma} \frac{-\not{p}_f + m_f}{p_f^2 - m_f^2} T_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a), \quad (\text{B.4})$$

---

<sup>6</sup>This identity is easily proven using a representation of the polarization vectors by Weyl spinors. Employing the conventions of Ref. [23], we have  $\varepsilon_+^{AB} = \varepsilon_+^\mu \sigma_\mu^{AB} = \sqrt{2} n^A k^B / \langle kn \rangle$  and  $\varepsilon_-^{AB} = \varepsilon_-^\mu \sigma_\mu^{AB} = \sqrt{2} k^A n^B / \langle kn \rangle^*$  for the polarization bispinors, so that

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} k_\rho n_\sigma &= \frac{i}{4} (\epsilon^{\dot{A}\dot{E}} \epsilon^{\dot{C}\dot{G}} \epsilon^{BD} \epsilon^{FH} - \epsilon^{\dot{A}\dot{C}} \epsilon^{\dot{E}\dot{G}} \epsilon^{BF} \epsilon^{DH}) \sigma_{AB}^\mu \sigma_{CD}^\nu \sigma_{EF}^\rho \sigma_{GH}^\sigma k_\rho n_\sigma \\ &= \frac{i}{4} (k^{\dot{A}} n^{\dot{C}} \epsilon^{BD} k_X n^X - \epsilon^{\dot{A}\dot{C}} k^B n^D k_{\dot{X}} n^{\dot{X}}) \sigma_{AB}^\mu \sigma_{CD}^\nu \\ &= \frac{i}{4} (-k^{\dot{A}} n^B n^{\dot{C}} k^D + n^{\dot{A}} k^B k^{\dot{C}} n^D) \sigma_{AB}^\mu \sigma_{CD}^\nu \\ &= \frac{i}{2} \langle kn \rangle \langle kn \rangle^* (\varepsilon_+^\mu \varepsilon_-^\nu - \varepsilon_-^\mu \varepsilon_+^\nu) = i(kn) (\varepsilon_+^\mu \varepsilon_-^\nu - \varepsilon_-^\mu \varepsilon_+^\nu). \end{aligned}$$

The only non-trivial step is the third equality which follows from a twofold application of Schouten's identity.

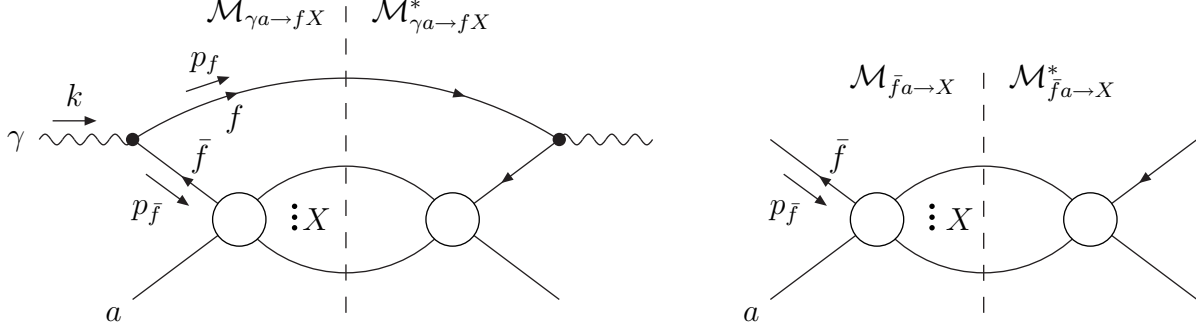


Figure 6: Generic squared diagram for the splitting  $\gamma \rightarrow f\bar{f}^*$  (left) and the corresponding squared diagram for the related process with an incoming  $\bar{f}$  (right).

where  $T_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a)$  includes all information of the subamplitude indicated by the open blob in Fig. 6 and  $\bar{T}_{\bar{f}a \rightarrow X} = (T_{\bar{f}a \rightarrow X})^\dagger \gamma_0$ . To leading order in  $m_f \rightarrow 0$ , the squared amplitude for the subprocess  $\bar{f}a \rightarrow X$  can be written as

$$|\mathcal{M}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a; \kappa_{\bar{f}})|^2 = \bar{T}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a) [\omega_{-\kappa_{\bar{f}}} \not{p}_{\bar{f}} + \mathcal{O}(m_f)] T_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a), \quad (\text{B.5})$$

with  $\omega_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$  and  $\kappa_{\bar{f}} = \pm$  denoting the sign of the  $\bar{f}$  helicity. In order to find the relation between these squared matrix elements, we insert identity (B.2) into Eq. (B.4) and eliminate the  $\epsilon$  tensor via Chisholm's identity,

$$i\epsilon^{\alpha\beta\gamma\delta}\gamma_\delta = (\gamma^\alpha\gamma^\beta\gamma^\gamma - g^{\alpha\beta}\gamma^\gamma + g^{\alpha\gamma}\gamma^\beta - g^{\beta\gamma}\gamma^\alpha)\gamma_5, \quad (\text{B.6})$$

i.e. we trade the  $\epsilon$  contributions for a  $\gamma_5$  insertion in the spinor chain. Next, we isolate the leading terms in the collinear limit  $kp_f = \mathcal{O}(m_f^2) \rightarrow 0$ . This limit can, e.g., be parametrized by the decomposition of the momentum of  $f$

$$p_f^\mu = (1-x)k^\mu + p_{f,\perp}^\mu + p_{f,r}^\mu \quad (\text{B.7})$$

with  $x = 1 - p_f^0/k^0$ ,  $kp_{f,\perp} = 0$ , and  $\mathbf{p}_{f,r} = \mathbf{0}$  (where boldface symbols refer the spatial parts of momenta). In this decomposition we have  $\mathcal{O}(p_{f,\perp}^0) = \mathcal{O}(p_{f,r}^0) = \mathcal{O}(m_f^2)$  and  $\mathbf{p}_{f,\perp}^2 = \mathcal{O}(m_f^2)$ . Thus, each component of the orthogonal 3-vector  $\mathbf{p}_{f,\perp}$  is of  $\mathcal{O}(m_f)$ . After some straightforward simplifying algebra, the result of applying the power counting to  $|\mathcal{M}_{\gamma a \rightarrow fX}|^2$  is<sup>7</sup>

$$|\mathcal{M}_{\gamma a \rightarrow fX}(k, p_a, p_f; \lambda_\gamma)|^2 \underset{kp_f \rightarrow 0}{\sim} \frac{Q_f^2 e^2}{2x(kp_f)} \bar{T}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a) \left\{ \left[ 1 - 2x(1-x) + \frac{xm_f^2}{kp_f} \right] \not{p}_{\bar{f}} \right.$$

<sup>7</sup>Actually there are also terms proportional to  $\lambda_\gamma m_f / (kp_f)^2 \bar{T}_{\bar{f}a \rightarrow X} \not{k} \not{p}_{f,\perp} \gamma_5 T_{\bar{f}a \rightarrow X}$ , which at first sight seem to contribute in  $\mathcal{O}(m_f^{-2})$  in the limit  $m_f \rightarrow 0$ . Although these terms obviously disappear from the subtraction function after setting  $m_f$  to zero, they potentially contribute to the corresponding integrated subtraction terms, in which the limit  $m_f \rightarrow 0$  is taken after the singular phase-space integration. However, the integration over the azimuthal angle of  $\mathbf{p}_f$ , which is always assumed in our analysis, leads to a further suppression by one power of  $m_f$ , so that the contribution to the phase-space integral of  $|\mathcal{M}_{\gamma a \rightarrow fX}|^2$  is mass suppressed. Thus, these terms are irrelevant in the construction of a subtraction function to separate mass-singular terms in the collinear cone.

$$\begin{aligned}
& -\lambda_\gamma \left[ 2x - 1 + \frac{m_f^2}{kp_f} \right] \gamma_5 \not{p}_{\bar{f}} \Big\} T_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a) \\
& \widetilde{kp_f \rightarrow 0} \frac{1}{2} Q_f^2 e^2 \left\{ (h_+^{\gamma f} + h_-^{\gamma f}) \left[ |\mathcal{M}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a; +)|^2 + |\mathcal{M}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a; -)|^2 \right] \right. \\
& \quad \left. + \lambda_\gamma (h_+^{\gamma f} - h_-^{\gamma f}) \left[ |\mathcal{M}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a; +)|^2 - |\mathcal{M}_{\bar{f}a \rightarrow X}(p_{\bar{f}}, p_a; -)|^2 \right] \right\}. \quad (\text{B.8})
\end{aligned}$$

The last form results from the last but one by simply substituting  $|\mathcal{M}_{\bar{f}a \rightarrow X}|^2$  and the  $h_\pm^{\gamma f}$  functions defined in Eq. (3.4), whose arguments are suppressed in the notation. This completes our proof of Eq. (3.3), which is a more compact version of this result.

In this section we have explicitly treated  $f$  as fermion and  $\bar{f}$  as antifermion. The opposite case with  $f$  being an antifermion and  $\bar{f}$  a fermion is obtained analogously and leads to the identical final result (3.3), although some signs in intermediate results are different. This fact is, of course, to be expected, because relations between squared helicity amplitudes cannot depend on our convention which fermion we call the antiparticle of the other.

## B.2 Dipole subtraction for $\gamma \rightarrow f\bar{f}^*$ splittings with massive final-state spectator

Here we give some details on the derivation of the integrated subtraction part presented in Section 3.3 for the collinear splitting  $\gamma \rightarrow f\bar{f}^*$  in the process  $\gamma a \rightarrow fjX$ , where  $j$  is a possibly massive spectator. We start by generalizing the form (3.24) of the new momenta upon restoring the correct dependence on  $m_f$ ,

$$\begin{aligned}
\tilde{p}_{\bar{f}}^\mu(x) &= \frac{\sqrt{\lambda_{fj,\gamma}}}{-\bar{P}^2} \left( xk^\mu + \frac{\bar{P}^2}{2P^2} P^\mu \right) - \frac{P^2 + m_f^2 - m_j^2}{2P^2} P^\mu, & \tilde{p}_{\bar{f}}^\mu &= \tilde{p}_{\bar{f}}^\mu(x_{fj,\gamma}), \\
\tilde{p}_j^\mu(x) &= P^\mu + \tilde{p}_{\bar{f}}^\mu(x), & \tilde{p}_j^\mu &= \tilde{p}_j^\mu(x_{fj,\gamma}),
\end{aligned} \quad (\text{B.9})$$

where the following shorthands are used,

$$\bar{P}^2 = P^2 - m_f^2 - m_j^2, \quad \lambda_{fj,\gamma} = \lambda(P^2, m_f^2, m_j^2), \quad (\text{B.10})$$

with the auxiliary function

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (\text{B.11})$$

The new momenta satisfy the on-shell conditions  $\tilde{p}_{\bar{f}}^2 = m_f^2$ ,  $\tilde{p}_j^2 = m_j^2$  and correctly behave in the collinear limit,  $\tilde{p}_{\bar{f}} \rightarrow xk$ , where  $kp_f = \mathcal{O}(m_f^2) \rightarrow 0$ . The splitting of the  $(N+1)$ -particle phase space into the corresponding  $N$ -particle phase space and the integral over the remaining singular degrees of freedom is given by

$$\int d\phi(p_f, p_j, k_X; k + p_a) = \int_0^{x_1} dx \int d\phi(\tilde{p}_j(x), k_X; \tilde{p}_{\bar{f}}(x) + p_a) \int [dp_f(P^2, x, z_{fj,\gamma})], \quad (\text{B.12})$$

with the explicit parametrization

$$\int [dp_f(P^2, x, z_{fj,\gamma})] = \frac{1}{2(2\pi)^3} \frac{-\bar{P}^2(p_a \tilde{p}_{\bar{f}}(x))}{x^2 s} \int_{z_1(x)}^{z_2(x)} dz_{fj,\gamma} \int d\phi_f, \quad (\text{B.13})$$

and  $k_X$  denoting the outgoing total momentum of  $X$ . The upper kinematical limit of the parameter  $x = x_{fj,\gamma}$  is given by

$$x_1 = \frac{-\bar{P}^2}{-\bar{P}^2 + 2m_fm_j}. \quad (\text{B.14})$$

The integration of the azimuthal angle  $\phi_f$  of  $f$  simply yields a factor  $2\pi$ , but the integration of the auxiliary parameter

$$z_{fj,\gamma} = \frac{kp_j}{kp_f + kp_j} \quad (\text{B.15})$$

with the boundary

$$z_{1,2}(x) = \frac{2m_j^2x + \bar{P}^2(x-1) \mp \sqrt{\bar{P}^4(1-x)^2 - 4m_f^2m_j^2x^2}}{2(P^2x - \bar{P}^2)} \quad (\text{B.16})$$

is non-trivial. The integration kernels occurring in the final result (3.25) are defined as

$$\mathcal{H}_{j,\tau}^f(P^2, x) = \frac{-\bar{P}^2}{2} \int_{z_1(x)}^{z_2(x)} dz_{fj,\gamma} h_{j,\tau}^{\gamma f}(k, p_f, p_j) \quad (\text{B.17})$$

and can be evaluated without problems analytically (even for finite  $m_f$ ) yielding Eq. (3.26) for  $m_f \rightarrow 0$ .

## C More details on the subtraction for $\gamma^* \rightarrow f\bar{f}$ splittings

In this appendix we supplement Section 4.3, where the subtraction procedure for collinear  $\gamma^* \rightarrow f\bar{f}$  splittings has been described for a final-state spectator  $j$ . In the following we fully take into account the spectator mass  $m_j$ . The derivation widely follows Ref. [8], where the treatment of the  $g^* \rightarrow Q\bar{Q}$  splitting with a massive quark  $Q$  has been considered. Our approach differs from the one described in Ref. [8] in the level of inclusiveness that is assumed in the collinear limit; in contrast to that paper we do not assume a recombination of the  $f\bar{f}$  pair in the collinear limit, but instead control the individual momentum flow of  $f$  and  $\bar{f}$ .

For arbitrary mass values  $m_f$  and  $m_j$  the subtraction function can be constructed as in Eq. (4.17) with the generalized radiator function

$$h_{f\bar{f},j}^{\mu\nu}(p_f, p_{\bar{f}}, p_j) = \frac{2}{(p_f + p_{\bar{f}})^2 v_j} \left[ -g^{\mu\nu} \left( 1 - 2\kappa \left( z_1 z_2 - \frac{m_f^2}{(p_f + p_{\bar{f}})^2} \right) \right) - \frac{4}{(p_f + p_{\bar{f}})^2} \left( z_{f\bar{f}j}^{(m)} p_f^\mu - \bar{z}_{f\bar{f}j}^{(m)} p_{\bar{f}}^\mu \right) \left( z_{f\bar{f}j}^{(m)} p_f^\nu - \bar{z}_{f\bar{f}j}^{(m)} p_{\bar{f}}^\nu \right) \right]. \quad (\text{C.1})$$

In addition to the parameters  $y_{f\bar{f}j}$  and  $z_{f\bar{f}j}$ , which are defined as in Eq. (4.19), we make use of the following auxiliary quantities,

$$\begin{aligned} \bar{P}^2 &= P^2 - 2m_f^2 - m_j^2, \\ v_f &= \sqrt{\frac{\bar{P}^2 y_{f\bar{f}j} - 2m_f^2}{\bar{P}^2 y_{f\bar{f}j} + 2m_f^2}}, \quad v_j = \frac{\sqrt{[2m_j^2 + \bar{P}^2(1 - y_{f\bar{f}j})]^2 - 4m_j^2 P^2}}{\bar{P}^2(1 - y_{f\bar{f}j})}, \\ z_{1,2} &= \frac{1}{2}(1 \mp v_j v_f), \quad z_{f\bar{f}j}^{(m)} = z_{f\bar{f}j} - \frac{1}{2}(1 - v_j), \quad \bar{z}_{f\bar{f}j}^{(m)} = \bar{z}_{f\bar{f}j} - \frac{1}{2}(1 - v_j). \end{aligned} \quad (\text{C.2})$$

The parameter  $\kappa$  is arbitrary, because the singular behaviour does not depend on it; in practice the independence of the final result on  $\kappa$  can be used as check. The auxiliary momenta entering the hard scattering matrix element for the subprocess  $ab \rightarrow \gamma j X$  also become more complicated,

$$\begin{aligned}\tilde{p}_j^\mu &= \frac{P^2 - m_j^2}{\sqrt{\lambda(P^2, (p_f + p_{\bar{f}})^2, m_j^2)}} \left( p_j^\mu - \frac{P p_j}{P^2} P^\mu \right) + \frac{P^2 + m_j^2}{2P^2} P^\mu, \\ \tilde{k}^\mu &= P^\mu - \tilde{p}_j^\mu, \quad P^\mu = p_f^\mu + p_{\bar{f}}^\mu + p_j^\mu.\end{aligned}\tag{C.3}$$

In order to integrate the subtraction function we need the azimuthal-averaged version of  $h_{f\bar{f},j}^{\mu\nu}$ ,

$$h_{f\bar{f},j}(p_f, p_{\bar{f}}, p_j) = \frac{2}{(p_f + p_{\bar{f}})^2 v_j} \left[ P_{f\gamma}(z_{f\bar{f}j}) + 2(1 - \kappa) z_1 z_2 + \frac{2\kappa m_f^2}{(p_f + p_{\bar{f}})^2} \right], \tag{C.4}$$

and an appropriate splitting of the phase space of the momenta  $p_f, p_{\bar{f}}, p_j$ ,

$$\begin{aligned}\int d\phi(p_f, p_{\bar{f}}, p_j; P) &= \int d\phi(\tilde{k}, \tilde{p}_j; P) \int [dp_f(P^2, y_{f\bar{f}j}, z_{f\bar{f}j})], \\ \int [dp_f(P^2, y_{f\bar{f}j}, z_{f\bar{f}j})] &= \frac{1}{4(2\pi)^3} \frac{\bar{P}^4}{P^2 - m_j^2} \int_{y_1}^{y_2} dy_{f\bar{f}j} (1 - y_{f\bar{f}j}) \int_{z_1(y_{f\bar{f}j})}^{z_2(y_{f\bar{f}j})} dz_{f\bar{f}j} \int d\phi_f,\end{aligned}\tag{C.5}$$

where

$$y_1 = \frac{2m_f^2}{\bar{P}^2}, \quad y_2 = 1 - \frac{2m_j(\sqrt{\bar{P}^2} - m_j)}{\bar{P}^2} \tag{C.6}$$

and  $z_{1,2}(y_{f\bar{f}j})$  are the  $z_{1,2}$  of Eq. (C.2), evaluated as functions of  $y_{f\bar{f}j}$ . Up to this point, the full dependence on  $m_f$  and  $m_j$  is kept.

Since we want to keep the momentum flow in the collinear limit open, i.e. the  $z_{f\bar{f}j}$  integration should be done numerically, we have to interchange the order of  $y_{f\bar{f}j}$  and  $z_{f\bar{f}j}$  integrations in the singular phase-space integration over  $\int [dp_f]$ . For arbitrary masses  $m_f$  and  $m_j$ , this seems hardly possible analytically, so that we focus on the limit  $m_f \rightarrow 0$  in the following, because this is the interesting case. We define

$$\begin{aligned}\mathcal{H}_{f\bar{f},j}(P^2, z) &= \frac{\bar{P}^2}{2} \int_{y_1(z)}^{y_2(z)} dy_{f\bar{f}j} (1 - y_{f\bar{f}j}) h_{f\bar{f},j}(p_f, p_{\bar{f}}, p_j), \\ H_{f\bar{f},j}(P^2) &= \int_0^1 dz \mathcal{H}_{f\bar{f},j}(P^2, z),\end{aligned}\tag{C.7}$$

where we were allowed to use  $m_f = 0$  in the prefactors and in the integration limits of  $z = z_{f\bar{f}j}$ . The relevant asymptotics of  $y_{1,2}(z)$  for  $m_f \rightarrow 0$  is

$$y_1(z) = \frac{m_f^2}{\bar{P}^2} \frac{z^2 + (1 - z)^2}{z(1 - z)}, \quad y_2(z) = \frac{\sqrt{4\bar{P}^2 z(1 - z) + m_j^2} - m_j}{\sqrt{4\bar{P}^2 z(1 - z) + m_j^2} + m_j}. \tag{C.8}$$

The actual integration over  $y_{f\bar{f}j}$  yields

$$\mathcal{H}_{f\bar{f},j}(P^2, z) = P_{f\gamma}(z) \left[ 2 \ln \left( \frac{\sqrt{4\bar{P}^2 z(1 - z) + m_j^2} - m_j}{2m_f} \right) - 1 - \eta(z) \right]$$

$$\begin{aligned}
& -2 \ln[1 - \eta(z)] + \frac{m_j^2[1 - \eta(z)]}{m_j^2 + \eta(z)\bar{P}^2} \Big] \\
& + \frac{2m_j^2}{\bar{P}^2} (1 - \kappa + z^2 + (1 - z)^2) \ln \left( 1 + \eta(z) \frac{\bar{P}^2}{m_j^2} \right) + 2z(1 - z) \quad (\text{C.9})
\end{aligned}$$

with

$$\eta(z) = \begin{cases} [1 - y_2(z)]z & \text{for } z < \frac{1}{2}, \\ [1 - y_2(z)](1 - z) & \text{for } z > \frac{1}{2}. \end{cases} \quad (\text{C.10})$$

The case  $m_j = 0$  given in Eq. (4.22) can be easily read off after realizing that  $\eta(z) = \mathcal{O}(m_j)$ . For the evaluation of  $H_{f\bar{f},j}(P^2)$  it is easier to integrate first over  $z$  and then over  $y_{f\bar{f},j}$ . The result is

$$H_{f\bar{f},j}(P^2) = \frac{4}{3} \ln \left( \frac{\sqrt{P^2} - m_j}{m_f} \right) - \frac{16}{9} + \frac{4m_j}{3(\sqrt{P^2} + m_j)} + \left( \kappa - \frac{2}{3} \right) \frac{2m_j^2}{\bar{P}^2} \ln \left( \frac{2m_j}{\sqrt{P^2} + m_j} \right), \quad (\text{C.11})$$

which could also be derived from Eq. (5.36) of Ref. [8]. For  $m_j = 0$  this obviously leads to the form given in Eq. (4.22).

## D More details on the subtraction for $f \rightarrow f\gamma^*$ splittings

### D.1 Factorization in the collinear limit

In this section we derive the asymptotic behaviour (5.2) of the squared amplitude  $|\mathcal{M}_{fa \rightarrow fX}|^2$  for the case where the incoming and outgoing light fermions become collinear. We consider polarized incoming fermions  $f$  with momentum  $p_f^\mu$  and helicity of sign  $\kappa_f = \pm$ . The corresponding Dirac spinor  $u(p_f, \kappa_f)$  is an eigenspinor of the helicity projector

$$\Sigma_{\kappa_f} = \frac{1}{2}(1 + \kappa_f \gamma_5 \not{p}_f), \quad (\text{D.1})$$

where the polarization vector

$$s_{p_f}^\mu = \left( \frac{|\mathbf{p}_f|}{m_f}, \frac{p_f^0}{m_f} \mathbf{e}_f \right) \quad (\text{D.2})$$

is aligned to the direction  $\mathbf{e}_f = \mathbf{p}_f/|\mathbf{p}_f|$  for helicity eigenstates. Defining the light-like vectors  $\tilde{k}^\mu = k_0(1, \mathbf{e}_f)$  and  $n^\mu = (1, -\mathbf{e}_f)$ , the polarization vector  $s_{p_f}^\mu$  can be decomposed into  $\tilde{k}^\mu$  and  $n^\mu$  as follows,

$$s_{p_f}^\mu = \frac{(p_f n)}{2m_f k_0} \tilde{k}^\mu - \frac{m_f}{2(p_f n)} n^\mu. \quad (\text{D.3})$$

Note that the momentum  $k^\mu$  of the virtual photon fulfills  $kn = \mathcal{O}(k^0)$  in the collinear limit, because then  $k^\mu = \tilde{k}^\mu + \mathcal{O}(m_f)$ . The vector  $n^\mu$  will be used as gauge vector in the explicit definition of photon polarization vectors for the subprocess  $\gamma a \rightarrow X$  below.

Power counting reveals that the logarithmic singularity arising from the phase-space region  $p_f p'_f = \mathcal{O}(m_f^2) \rightarrow 0$  ( $m_f \ll p_f$ ) originates from the square of diagrams in which the incoming fermion collinearly emits a photon that triggers the production of  $X$ . The generic form of such graphs is shown in Fig. 7. Assuming summation over the polarization

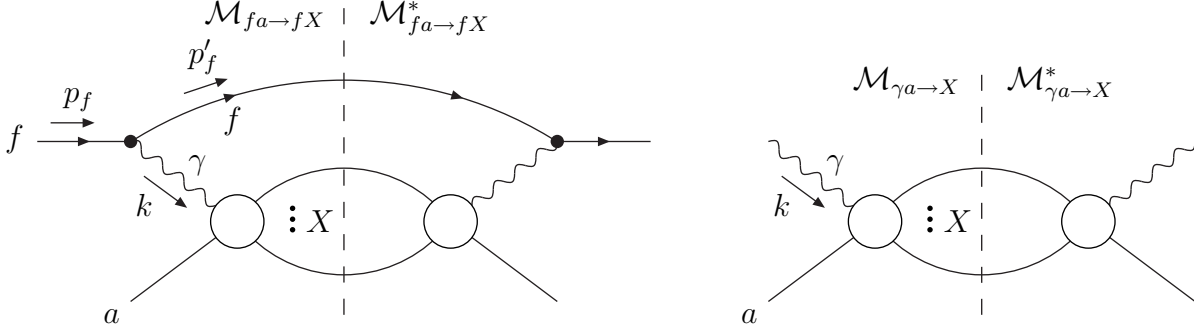


Figure 7: Generic squared diagram for the splitting  $f \rightarrow f\gamma^*$  (left) and the corresponding squared diagram for the related process with an incoming  $\gamma$  (right).

of the outgoing fermion  $f$ , the squared matrix element behaves like

$$|\mathcal{M}_{fa \rightarrow fX}(p_f, p_a, p'_f; \kappa_f)|^2 \underset{p_f p'_f \rightarrow 0}{\sim} \frac{N_{c,f} Q_f^2 e^2}{k^4} \text{Tr} \left\{ \Sigma_{\kappa_f} (\not{p}_f + m_f) \gamma_\mu (\not{p}'_f + m_f) \gamma_\nu \right\} T_{\gamma a \rightarrow X}^\mu(k, p_a)^* T_{\gamma a \rightarrow X}^\nu(k, p_a), \quad (\text{D.4})$$

where  $T_{\gamma a \rightarrow X}^\mu(k, p_a)$  includes all information of the subamplitude indicated by the open blob in Fig. 7. To leading order in  $m_f \rightarrow 0$ , the squared amplitude for the subprocess  $\gamma a \rightarrow X$  can be written as

$$|\mathcal{M}_{\gamma a \rightarrow X}(\tilde{k}, p_a; \lambda_\gamma)|^2 = \varepsilon_{\lambda_\gamma, \mu}^*(\tilde{k}) T_{\gamma a \rightarrow X}^\mu(\tilde{k}, p_a)^* \varepsilon_{\lambda_\gamma, \nu}(\tilde{k}) T_{\gamma a \rightarrow X}^\nu(\tilde{k}, p_a), \quad (\text{D.5})$$

with the helicity  $\lambda_\gamma = \pm$  of the incoming photon and the light-like vector  $\tilde{k}^\mu$ . In order to relate the  $fa$  process with the  $\gamma a$  subprocess, we first evaluate the trace in Eq. (D.4) and drop all terms that vanish owing to the Ward identity  $k_\mu T_{\gamma a \rightarrow X}^\mu(k, p_a) = 0$ . Inserting the form (D.3) of  $s_{p_f}^\mu$ , the result can be written as

$$|\mathcal{M}_{fa \rightarrow fX}(p_f, p_a, p'_f; \kappa_f)|^2 \underset{p_f p'_f \rightarrow 0}{\sim} \frac{N_{c,f} Q_f^2 e^2}{-k^2} \left\{ -g_{\mu\nu} - \frac{4p_{f,\mu} p_{f,\nu}}{k^2} - \frac{i\kappa_f}{k^2} \epsilon_{\mu\nu\alpha\beta} k^\alpha \left( \frac{(p_f n)}{k_0} \tilde{k}^\beta - \frac{m_f^2}{(p_f n)} n^\beta \right) \right\} \times T_{\gamma a \rightarrow X}^\mu(k, p_a)^* T_{\gamma a \rightarrow X}^\nu(k, p_a). \quad (\text{D.6})$$

Now we make use of the collinear limit which is characterized by  $p_f p'_f = m_f^2 - p_f k = \mathcal{O}(m_f^2) \rightarrow 0$ . We decompose the photon momentum according to

$$k^\mu = x p_f^\mu + k_\perp^\mu + k_r^\mu \quad (\text{D.7})$$

with  $x = k^0/p_f^0$ ,  $p_f k_\perp = 0$ , and  $\mathbf{k}_r = \mathbf{0}$ . In this decomposition we have  $\mathcal{O}(k_\perp^0) = \mathcal{O}(k_r^0) = \mathcal{O}(m_f^2)$  and  $\mathbf{k}_\perp^2 = \mathcal{O}(m_f^2)$ , i.e. the vector  $k_\perp^\mu$  can be counted as  $\mathcal{O}(m_f)$ . Moreover, we get  $k_\perp^2 = x^2 m_f^2 + k^2(1-x) + \mathcal{O}(m_f^4)$ . In the determination of the leading collinear behaviour of Eq. (D.6), we can replace the momentum  $k^\mu$  by the light-like momentum  $\tilde{k}^\mu = k^\mu + \mathcal{O}(m_f)$  in the two  $T_{\gamma a \rightarrow X}(k, p_a)$  terms. The expansion of the two terms with the  $\epsilon$ -tensor is also straightforward. With the help of identity (B.2), the contraction  $\epsilon_{\mu\nu\alpha\beta} k^\alpha n^\beta$  becomes

$$\epsilon_{\mu\nu\alpha\beta} k^\alpha n^\beta = \epsilon_{\mu\nu\alpha\beta} \tilde{k}^\alpha n^\beta + \mathcal{O}(m_f) = i(kn) \left[ \varepsilon_{+, \mu}(\tilde{k}) \varepsilon_{-, \nu}(\tilde{k}) - \varepsilon_{-, \mu}(\tilde{k}) \varepsilon_{+, \nu}(\tilde{k}) \right] + \mathcal{O}(m_f). \quad (\text{D.8})$$

The second contraction  $\epsilon_{\mu\nu\alpha\beta}k^\alpha\tilde{k}^\beta$  can be expanded upon writing  $\epsilon_{\mu\nu\alpha\beta} = g_\mu^{\mu'}g_\nu^{\nu'}g_\alpha^{\alpha'}\epsilon_{\mu'\nu'\alpha'\beta}$  with the following decomposition of the metric tensor,

$$g^{\mu\nu} = \frac{1}{2k_0}(n^\mu\tilde{k}^\nu + \tilde{k}^\mu n^\nu) - \varepsilon_+^\mu(\tilde{k})\varepsilon_-^\nu(\tilde{k}) - \varepsilon_-^\mu(\tilde{k})\varepsilon_+^\nu(\tilde{k}). \quad (\text{D.9})$$

The  $\epsilon$ -tensor now only appears as  $\epsilon_{\mu\nu\alpha\beta}\varepsilon_+^\mu(\tilde{k})\varepsilon_-^\nu(\tilde{k})n^\alpha\tilde{k}^\beta = 2ik_0$ , and the momentum  $\tilde{k}^\mu$  with an open index can be replaced via

$$\tilde{k}^\mu = \frac{2k_0}{(kn)} \left[ k^\mu + (\varepsilon_-(\tilde{k}) \cdot k) \varepsilon_+^\mu(\tilde{k}) + (\varepsilon_+(\tilde{k}) \cdot k) \varepsilon_-^\mu(\tilde{k}) \right] + \mathcal{O}(m_f^2), \quad (\text{D.10})$$

which follows from Eq. (D.9) upon contraction with  $k^\nu$ . This procedure spans the tensor  $\epsilon_{\mu\nu\alpha\beta}k^\alpha\tilde{k}^\beta$  in terms of  $\varepsilon_{\pm,\mu}(\tilde{k})\varepsilon_{\mp,\nu}(\tilde{k})$  and covariants involving  $k_\mu$  or  $k_\nu$ . The latter do not contribute because of the Ward identity  $k_\mu T_{\gamma a \rightarrow X}^\mu(k, p_a) = 0$ . The expansion of the various scalar products for  $m_f \rightarrow 0$  is straightforward, yielding

$$\begin{aligned} \epsilon_{\mu\nu\alpha\beta}k^\alpha\tilde{k}^\beta &= \frac{ik_0[k^2(x-2) - x^2m_f^2]}{x(p_f n)} \left[ \varepsilon_{+,\mu}(\tilde{k})\varepsilon_{-,\nu}(\tilde{k}) - \varepsilon_{-,\mu}(\tilde{k})\varepsilon_{+,\nu}(\tilde{k}) \right] \\ &+ (\text{terms proportional to } k_\mu \text{ or } k_\nu) + \mathcal{O}(m_f^3). \end{aligned} \quad (\text{D.11})$$

Inserting Eqs. (D.8) and (D.11) into Eq. (D.6) and performing the expansion in the collinear limit leads to the form given in Eqs. (5.2) and (5.3).

The final step of performing the azimuthal average around the collinear axis, which leads to Eqs. (5.5) and (5.6), is most easily carried out by fixing a specific coordinate frame. In a frame where the direction of  $\mathbf{p}_f$  is given by  $\mathbf{e}_f^T = (0, 0, 1)$ , the vectors  $\tilde{k}^\mu$  and  $\varepsilon_\pm^\mu(\tilde{k})$  are given by

$$\tilde{k}^\mu = (k_0, 0, 0, k_0), \quad \varepsilon_\pm^\mu(\tilde{k}) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0). \quad (\text{D.12})$$

Recall that  $k^\mu$  and  $\tilde{k}^\mu$  differ only by terms of  $\mathcal{O}(m_f)$  in the collinear limit. According to definition (D.7) the leading term of  $k_\perp^\mu$  takes the form

$$k_\perp^\mu = (0, -|\mathbf{k}_\perp| \cos \phi'_f, -|\mathbf{k}_\perp| \sin \phi'_f, 0) + \mathcal{O}(m_f^2), \quad (\text{D.13})$$

where  $\phi'_f$  is the azimuthal angle of  $\mathbf{p}'_f = \mathbf{p}_f - \mathbf{k}$ . In this parametrization the average  $\langle k_\perp^\mu k_\perp^\nu \rangle_{\phi'_f}$  is easily calculated to

$$\langle k_\perp^\mu k_\perp^\nu \rangle_{\phi'_f} = -\frac{k_\perp^2}{2} \text{diag}(0, 1, 1, 0) + \mathcal{O}(m_f^3) = -\frac{k_\perp^2}{2} E^{\mu\nu}(\tilde{k}) + \mathcal{O}(m_f^3), \quad (\text{D.14})$$

while it is trivially seen that the tensors  $\varepsilon_\pm^\mu(\tilde{k})\varepsilon_\pm^\nu(\tilde{k})^*$  do not change after taking the azimuthal average. With these considerations the transition from Eqs. (5.2) and (5.3) to the averaged form in Eqs. (5.5) and (5.6) is straightforward.



## D.2 Dipole subtraction for $f \rightarrow f\gamma^*$ splittings with massive final-state spectator

In Section 5.3 we have presented all formulas needed for the case of a massless final-state spectator in practice, but did not go into the details of their derivation. Here we close this gap by deriving the formalism in the more general situation of a possibly massive spectator  $j$ . We keep the general definition (5.22) of the subtraction function, but generalize the subtraction kernel as follows,

$$h_{j,\kappa_f}^{ff,\mu\nu}(p_f, p'_f, p_j) = \frac{-1}{(p_f - p'_f)^2} \left[ -g^{\mu\nu} - \frac{4(1-x)}{x^2} \frac{\tilde{k}_\perp^\mu \tilde{k}_\perp^\nu}{\tilde{k}_\perp^2} \frac{(p_f - p'_f)^2(1-x) + m_f^2 x^2}{(p_f - p'_f)^2(1-x)} \right. \\ \left. + \frac{\kappa_f}{x} \left( 2 - x + \frac{2x^2 m_f^2}{(p_f - p'_f)^2} \right) \left( \varepsilon_+^\mu(\tilde{k})^* \varepsilon_+^\nu(\tilde{k}) - \varepsilon_-^\mu(\tilde{k})^* \varepsilon_-^\nu(\tilde{k}) \right) \right], \quad (\text{D.15})$$

because we need the correct dependence on the emitter mass  $m_f$  for the integration of  $|\mathcal{M}_{\text{sub}}|^2$  below. The auxiliary parameters still have the form (5.24), but the auxiliary momenta become more complicated,

$$\tilde{k}^\mu(x) = \frac{m_j^2 - P^2}{-\bar{P}^2} \frac{x}{R(x)} \left( p_f^\mu + \frac{\bar{P}^2 + 2m_f^2 x}{2xP^2} P^\mu \right) + \frac{m_j^2 - P^2}{2P^2} P^\mu, \quad \tilde{k}^\mu = \tilde{k}^\mu(x_{fj,f}), \\ \tilde{p}_j(x) = P^\mu + \tilde{k}^\mu(x), \quad \tilde{p}_j = \tilde{p}_j(x_{fj,f}), \\ \tilde{k}_\perp^\mu = \frac{p_j \tilde{k}}{\tilde{p}_j \tilde{k}} p_f'^\mu - \frac{p_f' \tilde{k}}{\tilde{p}_j \tilde{k}} p_j^\mu. \quad (\text{D.16})$$

Here we made use of the abbreviations

$$P^\mu = p_f'^\mu + p_j^\mu - p_f^\mu, \quad \bar{P}^2 = P^2 - 2m_f^2 - m_j^2, \quad R(x) = \frac{\sqrt{(\bar{P}^2 + 2m_f^2 x)^2 - 4x^2 m_f^2 P^2}}{-\bar{P}^2}, \quad (\text{D.17})$$

The new momenta satisfy the on-shell conditions  $\tilde{k}^2 = 0$ ,  $\tilde{p}_j^2 = m_j^2$  and correctly behave in the collinear limit,  $\tilde{k} \rightarrow xp_f$ , where  $p_f p_f' = \mathcal{O}(m_f^2) \rightarrow 0$ . Before carrying out the singular integrations, we average the subtraction function over  $\phi'_f$ , yielding

$$\langle |\mathcal{M}_{\text{sub}}(\kappa_f)|^2 \rangle_{\phi'_f} = N_{c,f} Q_f^2 e^2 h_{j,\tau}^{ff}(p_f, p'_f, p_j) |\mathcal{M}_{\gamma a \rightarrow jX}(\tilde{k}, p_a; \lambda_\gamma = \tau \kappa_f)|^2 \quad (\text{D.18})$$

with summation over  $\tau = \pm$  and

$$h_{j,\tau}^{ff}(p_f, p'_f, p_j) = \frac{-1}{\bar{P}^2 z_{fj,f} + 2m_f^2 x_{fj,f}} \left[ P_{\gamma f}(x_{fj,f}) + \frac{2m_f^2 x_{fj,f}^2}{\bar{P}^2 z_{fj,f} + 2m_f^2 x_{fj,f}} \right. \\ \left. + \tau \left( 2 - x_{fj,f} + \frac{2x_{fj,f}^3 m_f^2}{\bar{P}^2 z_{fj,f} + 2m_f^2 x_{fj,f}} \right) \right]. \quad (\text{D.19})$$

The splitting of the  $(N+1)$ -particle phase space into the corresponding  $N$ -particle phase space and the integral over the remaining singular degrees of freedom is given by

$$\int d\phi(p'_f, p_j, k_X; p_f + p_a) = \int_0^{x_1} dx \int d\phi(\tilde{p}_j(x), k_X; \tilde{k}(x) + p_a) \int [dp'_f(P^2, x, z_{fj,f})], \quad (\text{D.20})$$

with the explicit parametrization

$$\int [dp'_f(P^2, x, z_{fj,f})] = \frac{1}{4(2\pi)^3} \frac{\tilde{s}}{\bar{s}} \frac{-\bar{P}^2}{x^2 R(x)} \int_{z_1(x)}^{z_2(x)} dz_{fj,f} \int d\phi'_f. \quad (\text{D.21})$$

The upper kinematical limit of the parameter  $x = x_{fj,f}$  is given by

$$x_1 = \frac{-\bar{P}^2}{-\bar{P}^2 + 2m_f m_j}. \quad (\text{D.22})$$

The integration of the azimuthal angle  $\phi'_f$  of  $f(p'_f)$  simply yields a factor  $2\pi$ . The non-trivial integration over  $z_{fj,f}$  has the boundary

$$z_{1,2}(x) = \frac{2m_f^2 x + \bar{P}^2(x-1) \mp R(x) \sqrt{\bar{P}^4(1-x)^2 - 4m_f^2 m_j^2 x^2}}{2[\bar{P}^2(x-1) + (m_f^2 + m_j^2)x]}. \quad (\text{D.23})$$

Defining the integrated subtraction kernel according to

$$\mathcal{H}_{j,\tau}^{ff}(P^2, x) = \frac{-\bar{P}^2}{2R(x)} \int_{z_1(x)}^{z_2(x)} dz_{fj,f} h_{j,\tau}^{ff}(p_f, p'_f, p_j), \quad (\text{D.24})$$

the cross-section contribution of the subtraction part takes the form (5.26) in the limit  $m_f \rightarrow 0$ . For a non-zero spectator mass  $m_j$ , the function  $\mathcal{H}_{j,\tau}^{ff}(P^2, x)$  reads

$$\mathcal{H}_{j,\tau}^{ff}(P^2, x) = \frac{1}{2} \ln \left( \frac{\bar{P}^4(1-x)^2}{x^3 m_f^2 [-\bar{P}^2(1-x) + m_j^2 x]} \right) \left[ P_{\gamma f}(x) + \tau(2-x) \right] - \frac{1-x}{x} - \tau(1-x). \quad (\text{D.25})$$

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